Conjugacy Classes and Geodesic Loops (Lecture 35)

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Let X be a path connected topological space and let $f : S^1 \to X$ be a map. Then f determines a conjugacy class $[f]$ in the fundamental group $\pi_1 X$. Our goal in this lecture is to show any nonzero conjugacy class is represented by an essentially *canonical* map f in the case where X is a hyperbolic surface.

Lemma 1. *Assume that X is a compact Riemannian manifold. Then any conjugacy class $\gamma \in \pi_1 X$ can be represented by a closed geodesic $f : S^1 \to X$.*

Proof. Endow the circle S^1 with its standard Riemannian metric, normalized so that the circle has total length 1. Define the *Lipschitz constant* $L(f)$ of a loop f to be the supremum of

$$\frac{d(f(x), f(y))}{d(x, y)}$$

. This supremum may be infinite: however, for a smooth path f it is finite (and coincides with maximum length of the derivative f' on S^1). Let c be the infimum of the set $\{L(f)\}$, where f varies over all representatives of γ. We will show that this infimum is achieved: that is, there exists a loop f with $L(f) = c$. Then f must be a smooth geodesic (of speed c) if it fails to be a geodesic near some point t, we can obtain a shorter loop representing γ by modifying f near t (and then changing our parametrization).

To prove that c is achieved, choose a sequence of loops $\{f_i\}_{i \geq 0}$ such that the real numbers $L(f_i)$ converge to c from above. Passing to a subsequence, we may assume that $L(f_i) < c + 1$. Choose a countable dense subset $\{t_j\} \subseteq S^1$. Since X is compact, we can pass to a subsequence and thereby assume that $f_0(t_0), f_1(t_0), \ldots$ converges to some point $x_0 \in X$. Similarly, we can pass to a subsequence of $\{f_1, f_2, \ldots\}$ and thereby guarantee that the sequence $f_1(t_1), f_2(t_1), \ldots$ converges to a point $x_1 \in X$. Proceeding in this way, we obtain a refinement of the original sequence such that $\{f_i(t_j)\}_{i \geq 0}$ converges to some $x_j \in X$. We define a new map $f : \{t_j\} \to X$ by the formula $f(t_j) = x_j$. We claim that f extends to a continuous map $S^1 \to X$ having $L(f) \leq c$. To prove this, it suffices to show that

$$d(f(t_i), f(t_j)) \leq c d(t_i, t_j)$$

for each pair of integers $i \neq j$. This is clear:

$$d(f(t_i), f(t_j)) \leq d(f(t_i), f_n(t_i)) + d(f(t_j), f_n(t_j) + d(f_n(t_i), f_n(t_j)) \leq \epsilon + L(f_n) d(t_i, t_j)$$

where ϵ can be made arbitrarily small (by choosing n large enough) and $L(f_n)$ can be made arbitrarily close to c.

Choose $\epsilon > 0$ small enough that every pair of points of X within a distance ϵ are connected by a unique geodesic. For $n \gg 0$, we have $d(f(t), f_n(t)) < \epsilon$ for all t, so that f and f_n can be connected by a geodesic homotopy; it follows that f is homotopic to f_n and therefore represents the free homotopy class γ. □

Let us now suppose that X is a hyperbolic surface, so that X can be represented as H/Γ where H is the upper half place $\{x + iy : y > 0\}$ and Γ is a group which acts on H by hyperbolic isometries. Then

$\Gamma \simeq \pi_1 X$, and we can identify Γ with a subgroup of the group $P\,\mathrm{SL}_2(\mathbb{R})$ of linear fractional transformations of the form

$$z \mapsto \frac{az+b}{cz+d}.$$

It is traditional to decompose elements of $P\,\mathrm{SL}_2(\mathbb{R})$ into three types:

(i) An element $A \in \mathrm{SL}_2(\mathbb{R})$ is called *elliptic* if $|\operatorname{tr}(A)| < 2$. In this case, the eigenvalues of A are unit complex numbers (and complex conjugate to one another); the transformation A itself is given by $z \mapsto \frac{\cos(\theta)z - \sin(\theta)}{\sin(\theta)z + \cos(\theta)}$ for some real number θ. Elliptic elements never appear in the discrete groups Γ under consideration, because they always have fixed points in the upper half plane (the above transformation has the complex number $z = i$ as a fixed point).

(ii) An element $A \in \mathrm{SL}_2(\mathbb{R})$ is called *parabolic* if $|\operatorname{tr}(A)| = 2$; in this case, the eigenvalues of A are both ± 1 but A is generally not semisimple: it is conjugate to a transformation of the form $z \mapsto z + t$ for some real number t. Nontrivial transformations of this kind cannot appear in Γ when the quotient $X = H/\Gamma$ is compact. For suppose otherwise: then, by Lemma 1, we would have a geodesic loop $f : S^1 \to X$ representing the conjugacy class of a parabolic transformation $z \mapsto z + t$. Then f lifts to a geodesic path $\widetilde{f}$ with the translation-invariance property $\widetilde{f}(x+1) = \widetilde{f}(t)$. There is no geodesic in the upper half plane with this property: the unique geodesic passing through $\widetilde{f}(0)$ and $\widetilde{f}(0)+t$ does not pass through $\widetilde{f}(0) + 2t$.

This argument does not apply if the quotient H/Γ is noncompact. In fact, a finite volume quotient H/Γ is compact if and only if Γ contains no parabolic elements: in fact, there is a bijection between cusps of H/Γ and conjugacy classes of maximal parabolic subgroups of Γ.

(iii) An element $A \in \mathrm{SL}_2(\mathbb{R})$ is called *hyperbolic* if $|\operatorname{tr}(A)| > 2$ (modifying A by a sign, we may assume that $\operatorname{tr}(A) > 2$). In this case, A has distinct real eigenvalues $\lambda, \frac{1}{\lambda}$ for some $\lambda > 1$. Then A is conjugate to the transformation $z \mapsto \lambda z$. In this case, there is a unique geodesic path $\widetilde{f} : \mathbb{R} \to H$ satisfying $\widetilde{f}(t+1) = A\widetilde{f}(t)$: namely, the path given by the formula $\widetilde{f}(t) = \lambda^t i$. This path descends to a geodesic loop $f : S^1 \to H/\Gamma$ representing the conjugacy class of $\pm A$ in $\Gamma \simeq \pi_1 H/\Gamma$.

The above analysis proves the following result:

Theorem 2. *Let $X = H/\Gamma$ be a compact hyperbolic surface. Then every nontrivial element γ of $\pi_1 X \simeq \Gamma \subseteq P\,\mathrm{SL}_2(\mathbb{R})$ is hyperbolic. Moreover, the conjugacy class of γ can be represented by a geodesic loop $f : S^1 \to X$ which is* unique *up to reparametrization.*

In other words, if X is a hyperbolic surface, then every conjugacy class in $\pi_1 X$ has a canonical representative. We now show that these representatives are well-behaved:

Theorem 3. *Let X be a hyperbolic surface, and suppose we are given distinct nontrivial conjugacy classes $\gamma_1, \ldots, \gamma_n \in \pi_1 X$. The following conditions are equivalent:*

(1) *The conjugacy classes γ_i can be represented by simple closed curves $C_i \subseteq X$ such that $C_i \cap C_j = \emptyset$ for $i \neq j$.*

(2) *The canonical geodesic representatives for $\gamma_1, \ldots, \gamma_n$ are simple closed curves $C_i \subseteq X$ such that $C_i \cap C_J = \emptyset$ for $i \neq j$.*

Proof. It is clear that (2) $\Rightarrow$ (1). Suppose that (1) is satisfied. Let $\{f_i : S^1 \to X\}_{1 \leq i \leq n}$ be a parametrizations of the curves C_i which satisfy condition of (1), and let $\{g_i : S^1 \to X\}_{1 \leq i \leq n}$ be the geodesic representatives of the conjugacy classes γ_i. We wish to prove that each g_i is a simple curve, and that $g_i(S^1) \cap g_j(S^1) = \emptyset$ for $i \neq j$. We will prove the latter; the former follows by the same argument.

Choose a lifting of g_i to a geodesic path $\widetilde{g}_i : \mathbb{R} \to D$, where D is the unit disk. If $g_i(S^1) \cap g_j(S^1) \neq \emptyset$, then we can lift g_j to a geodesic path $\widetilde{g}_j : \mathbb{R} \to D$ such that $\widetilde{g}_i(\mathbb{R})$ and $\widetilde{g}_j(\mathbb{R})$ intersect. Let $a, b \in \partial D$ be the endpoints of $\widetilde{g}_i$ on the circle at infinity, and let a', b' be the endpoints of $\widetilde{g}_j$. Note that $\widetilde{g}_i(\mathbb{R})$ and $\widetilde{g}_j(\mathbb{R})$ intersect if and only if the sets $\{a, b\}$ and $\{a', b'\}$ are disjoint, and the points a' and b' belong to different components of $\partial D - \{a, b\}$.

Since f_i and g_i represent the same conjugacy class in $\pi_1 X$, there is a homotopy h from f_i to g_i. Lifting this homotopy to the universal cover, we get a lift $\widetilde{f}_i : \mathbb{R} \to D$ of f_i and a homotopy from $\widetilde{f}_i$ to $\widetilde{g}_i$. This homotopy moves points by a bounded amount with respect to the hyperbolic metric on D. Consequently, it moves points which are close to the boundary ∂D by very small amounts with respect to the Euclidean metric on the closure of D. It follows that $\widetilde{f}_i$ has the same endpoints a and b as $\widetilde{g}_i$.

A similar argument shows that we can lift f_j to a path $\widetilde{f}_j : \mathbb{R} \to D$ having endpoints $a', b' \in \partial D$. If a' and b' belong to different components of $\partial D - \{a, b\}$, then $\widetilde{f}_i(\mathbb{R})$ and $\widetilde{f}_j(\mathbb{R})$ must have a point of intersection $\widetilde{x} \in D$. The image of $\widetilde{x}$ is a point $x \in f_i(S^1) \cap f_j(S^1) \subseteq X$, contradicting our assumptions. $\square$