Existence of Prime Decompositions (Lecture 25)

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In this lecture, we begin our study of 3-manifolds. Our ultimate goal is to say something about the classification of 3-manifolds. To this end, we begin by considering an arbitrary compact 3-manifold M: what might it look like?

We observe that M can be written as a disjoint union of finitely many connected 3-manifold. Consequently, there is no harm in assuming (as we will from now on) that all of our 3-manifolds M are connected, so that $\pi_0 M \simeq *$. Consider now the fundamental group $\pi_1 M$. Let $\widetilde{M}$ denote the universal cover of M. If $\pi_1 M$ is finite, then $\widetilde{M}$ is a compact, simply connected 3-manifold. In this case, the structure of M is understood:

Theorem 1 (Perelman; Poincare Conjecture)**.** *Let $\widetilde{M}$ be a simply connected compact 3-manifold. Then $\widetilde{M} \simeq S^3$.*

The manifold M itself can be recovered as a quotient $S^3/\pi_1 M$, for some free action of the finite group $\pi_1 M$ on the 3-sphere S^3. There are a number of possibilities for what such an action can look like (for example, Lens spaces can be obtained via this construction); we will return to this point in a later lecture. For present purposes, we will regard these examples as "understood" and move on the case where the fundamental group $\pi_1 M$ is infinite.

If $\pi_1 M$ is infinite, the universal cover $\widetilde{M}$ is noncompact. It follows that $\mathrm{H}_3(\widetilde{M};\mathbf{Z}) \simeq \mathrm{H}^0_c(\widetilde{M};\mathbf{Z}) \simeq 0$, by Poincare duality. Since $\widetilde{M}$ is a simply connected space of dimension 3, we have two possibilties:

(i) The second homology group $\mathrm{H}_2(\widetilde{M};\mathbf{Z})$ does not vanish. By the Hurewicz theorem, this group is isomorphic to $\pi_2 \widetilde{M} \simeq \pi_2 M$, so that there are nontrivial maps $S^2 \to M$.

(ii) The universal cover $\widetilde{M}$ is contractible, so that $M = \widetilde{M}/\pi_1(M)$ is homotopy equivalent to the classifying space $B\pi_1 M$.

Our goal in the next few lectures is to show that the study of 3-manifolds in general can be reduced to the case (ii). As a first step, we consider the prototypical example of 3-manifolds M which do *not* satisfy (ii). Let M_0 and M_1 be a pair of 3-manifolds containing points x and y. Let M_0' and M_1' denote the 3-manifolds with boundary S^2 obtained by removing small balls around x and y (or by performing real blow-ups at x and y). We denote the amalgam $M_0' \coprod_{S^2} M_1'$ by $M_0 \# M_1$; this 3-manifold is called the *connect sum of* M_0 *and* M_1.

Warning 2. The connect sum $M_0 \# M_1$ depends not only on M_0 and M_1, but on a choice of identification of the boundaries $\partial M_0' \simeq S^2 \simeq \partial M_1'$. This choice of identification only matters up to isotopy (if we are interested only in the diffeomorphism class of the connect sum $M_0 \# M_1$), but the space $\mathrm{Diff}(S^2) \simeq O(3)$ has two different connected components, as we saw in the last lecture. Note however that if M_0 and M_1 are oriented, then there is a unique isotopy class of identifications such that $M_0 \# M_1$ admits an orientation compatible with those of M_0 and M_1. For simplicity, we will restrict our attention to the oriented case.

The operation $\#$ is commutative and associative up to diffeomorphism. Moreover, it has a unit given by the 3-sphere S^3: we have $S^3 \# M \simeq M$ for any 3-manifold M.

Definition 3. Let M be a 3-manifold which is not a 3-sphere. We say that M is *prime*, for any decomposition $M \simeq M_0 \# M_1$, either M_0 or M_1 is diffeomorphic to S^3.

Our goal in this lecture is to prove the following:

Theorem 4. *Let M be a 3-manifold. Then M admits a decomposition*

$$M \simeq M_1 \# M_2 \# \cdots \# M_n$$

where each M_i is prime. (Here we allow the degenerate possibility that $n = 0$, in which case the expression on the right side means the 3-manifold S^3.)

In the next lecture, we will prove a theorem of Milnor which asserts that the prime factors M_i of M are unique up to diffeomorphism. For the moment, we will be content to prove the existence of a prime factorization asserted by Theorem 4.

Notation 5. Let M be a compact 3-manifold. The fundamental group $\pi_1 M$ is a finitely generated group. We let $n(M)$ denote the minimal number of generators for $\pi_1 M$. Note that $n(M) = 0$ if and only if $\pi_1 M \simeq *$, which (by virtue of the Poincare conjecture) is equivalent to the assertion that M is a 3-sphere.

We will prove Theorem 4 using induction on $n(M)$. If $n(M) = 0$, then $M \simeq S^3$ and there is nothing to prove. Similarly, if M is prime then we are done. Otherwise, we can write $M \simeq M' \# M''$ where M' and M'' are not diffeomorphic to S^3, so that $n(M'), n(M'') > 0$. If M' and M'' admit prime factorizations, then these together give a prime factorization of M. The existence of these prime factorizations follows immediately from the inductive hypothesis and the following:

Lemma 6. *For any pair of compact 3-manifolds M' and M'', we have $n(M' \# M'') = n(M') + n(M'')$.*

Remark 7. The proof of Theorem 4 sketched above depends on the Poincare conjecture. However, Theorem 4 was known long before the Poincare conjecture. To give a proof independent of the Poincare conjecture, special considerations are needed to show the existence of prime factorizations when $n(M) = 0$. We will not pursue the point further here.

To prove Lemma 6 we observe that since S^2 is simply connected, van Kampen's theorem implies that $\pi_1(M' \# M'')$ is the free product $\pi_1 M' \star \pi_1 M''$ of the groups M' and M''. The inequality $n(M' \# M'') \leq n(M') + n(M'')$ is obvious, since any if $\{g_i\}$ is a collection of generators for $\pi_1 M'$ and $\{h_j\}$ is a collection of generators for $\pi_1 M''$, then $\{g_i, h_j\}$ is a collection of generators for $\pi_1 M' \star \pi_1 M''$. The reverse inequality follows from the following:

Theorem 8 (Grushko)**.** *Let F be a finitely generated free group, and let $\phi : F \to G \star H$ be a surjection of groups. Then F can be decomposed as a free product $F_0 \star F_1$ so that ϕ is a free product of maps $\phi_0 : F_0 \to G$, $\phi_1 : F_1 \to H$.*

Remark 9. In the situation of Theorem 8, the groups F_0 and F_1 are automatically free (since they are subgroups of the free group F) and finitely generated (since the rank of F is the sum of the ranks of F_0 and F_1). Since ϕ is surjective, ϕ_0 and ϕ_1 are also surjective, so that $n(G) + n(H) \leq n(F_0) + n(F_1) = n(F)$, where $n(X)$ denotes the minimal number of generators for a group X.

We will describe a geometric proof of Grushko's theorem, due to Stallings. Let BG and BH denote classifying spaces for G and H, and consider the space $X = BG \vee BH$ obtained by gluing BG and BH together along a point which we will denote by $*$. By van Kampen's theorem we have $\pi_1 X = G \star H$. In fact, X is a classifying space $B(G \star H)$, though we will not need to know this.

Choose a system of generators $\{v_1, \ldots, v_k\}$ for the group F. We regard ϕ as a map $F \to \pi_1(X, *)$, so that each $\phi(v_i)$ is represented by a loop L_i from $*$ to itself in X. Without loss of generality, we can assume that L_i is a composition of finitely many loops $L_i = L_{i,0} \circ \ldots \circ L_{i,n_i}$ where each L_{i,n_i} belongs entirely to BG or to BH. Let Y denote the bouquet of circles $\vee_i S^1$, so that the maps $\{L_i\}_{1 \leq i \leq k}$ determine a continuous map $f : Y \to X$. Using the above formulas for L_i, we conclude that Y can be written as a union of subgraphs

$$Y_G \coprod_{Y_0} Y_H$$

where $f(Y_G) \subseteq BG$, $f(Y_H) \subseteq BH$, Y_0 is a finite number of points, and $f(Y_0) = \{*\}$.

To prove Theorem 8, we will construct the following:

(1) A homotopy equivalence $Y \hookrightarrow K$.

(2) A decomposition $K \simeq K_G \coprod_{K_0} K_H$ extending the decomposition $Y \simeq Y_G \coprod_{Y_0} Y_H$, where the topological space K_0 is a graph without loops (in other words, a union of finitely many trees) and therefore homotopy equivalent to finitely many points.

(3) A map $f' : K \to X$ extending f, which carries K_G into BG, K_H into BH, and K_0 to $*$.

so that the following condition is satisfied:

(4) The space K_0 is a tree.

Then $F \simeq \pi_1 Y \simeq \pi_1 K$ by (1), and the map $\phi : F \to G \star H$ can be identified with $f'_* : \pi_1 K \to \pi_1 X$ by (3). Using (4) and van Kampen's theorem, we deduce that $F = \pi_1 K \simeq \pi_1 K_G \star \pi_1 K_H$, and we will have the desired decomposition of F.

If Y_0 is connected, we can take $K = Y$ and there is nothing to prove. In the general case, we proceed in several steps. We first show that it is possible to construct the data described in (1), (2), and (3) so that the following weaker version of condition (4) holds:

(4′) There exist two different connected components of Y_0 which belong to the same component of K_0.

If we can satisfy this condition, we then replace Y by K and repeat the same argument. The cardinality of the sets $\pi_0 K_0$ will form a decreasing chain as we proceed, and must eventually stabilize to the case where K_0 is connected (and therefore a tree, by virtue of (2).

Let $C_1, \ldots, C_m$ denote the set of path components of Y_0, and choose a point $y_i \in C_i$ for $1 \leq i \leq m$. Since Y is path connected, we can choose a path γ in Y from y_1 to y_2. Note that $f(\gamma)$ is a loop in X based at the point $*$, and therefore represents an element of $\pi_1 X$. Since ϕ is surjective, we can compose the original path γ with a loop based at y_1, and thereby arrange that $f(\gamma)$ is nullhomotopic. We have a homotopy

$$\gamma \simeq \gamma_1 \circ \ldots \circ \gamma_p$$

where each of the paths γ_i is supported entirely in Y_G or in Y_H and has endpoints in $\{y_1, \ldots, y_m\}$. We may assume that if any path γ_a begins and ends at the same point y_j, then $f(\gamma_a)$ is not nullhomotopic: otherwise, we can replace γ by the path

$$\gamma_1 \circ \ldots \circ \gamma_{a-1} \circ \gamma_{a+1} \circ \ldots \circ \gamma_p.$$

Concatenating the paths γ_a if necessary (and possibly swapping G with H), we can assume that γ_a is a path in Y_G when a is odd and a path in Y_H when a is even. We observe that each $f(\gamma_a)$ is a closed loop in X, so we have

$$1 = [f(\gamma_1)] \ldots [f(\gamma_p)] \in \pi_1 X \simeq G \star H.$$

Using the structure of the free product $G \star H$, we deduce that some factor $[f(\gamma_a)]$ must vanish. Without loss of generality, we may assume that γ_a is a path in Y_G from y_i to y_j; since $f(\gamma_a)$ is nullhomotopic we have $i \neq j$. Note that since the map $G \to G \star H$ is injective, the map $f(\gamma_a)$ is already nullhomotopic as a map in BG.

Let K_0 be the space obtained from Y_0 by adjoining a path τ from y_j to y_i, let $K_H = K_0 \coprod_{Y_0} Y_H$. We now let K_G be the space obtained from $K_0 \coprod_{Y_0} Y_G$ by attaching a 2-cell bounding the loop $\tau \circ \gamma_a$. Since $f(\gamma_a)$ is nullhomotopic in BG, we can extend f to a map $f'_G : K_G \to BG$ which takes the constant value $*$ on the path τ. Then f'_G and f determine a map $f' : K \to X$, which is easily seen to satisfy (1), (2), (3), and (4′).