Piecewise Linear Topology (Lecture 2)

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Our main goal for the first half of this course is to discuss the relationship between smooth manifolds and piecewise linear manifolds. In this lecture, we will set the stage by introducing the essential definitions.

Definition 1. Throughout these lectures, we will use the term *manifold* to refer to a paracompact Hausdorff space M with the property that each point $x \in M$ has an open neighborhood homeomorphic to $\mathbb{R}^n$, for some fixed integer $n \geq 0$; we refer to n as the *dimension* of M.

The study of manifold topology becomes substantially easier if we assume that our manifolds are endowed with additional structures, such as a smooth structure.

Definition 2. Let M be a manifold. We let $\mathcal{O}_M^{\mathrm{Top}}$ denote the sheaf of continuous real-valued functions on M, so that for each open set $U \subseteq M$ we have $\mathcal{O}_M^{\mathrm{Top}}(U) = \{f : U \to \mathbb{R} : f \text{ is continuous}\}$.

A *smooth structure* on M consists of a subsheaf $\mathcal{O}_M^{\mathrm{sm}} \subseteq \mathcal{O}_M^{\mathrm{Top}}$ with the following property: for every point $x \in M$, there exists an open embedding $f : \mathbb{R}^n \to M$ whose image contains f, such that $f^* \mathcal{O}_M^{\mathrm{sm}} \subseteq \mathcal{O}_{\mathbb{R}^n}^{\mathrm{Top}}$ can be identified with the sheaf of smooth (in other words, infinitely differentiable) functions on $\mathbb{R}^n$. In this case, we will refer to f as a *smooth chart* on M.

A *smooth manifold* is a manifold M equipped with a smooth structure. If $f : M \to N$ is a continuous map between smooth manifolds, we will say that f is *smooth* if the map $f^* \mathcal{O}_N^{\mathrm{sm}} \to \mathcal{O}_M^{\mathrm{Top}}$ factors through $\mathcal{O}_M^{\mathrm{sm}}$: in other words, if and only if composition with f carries smooth functions on N to smooth functions on M.

We now introduce the (perhaps less familiar) notion of a *piecewise linear*, or *combinatorial* manifold.

Definition 3. Let K be a subset of a Euclidean space $\mathbb{R}^n$. We will say that K is a *linear simplex* if it can be written as the convex hull of a finite subset $\{x_1, \ldots, x_k\} \subset \mathbb{R}^n$ which are independent in the sense that if $\sum c_i x_i = 0 \in \mathbb{R}^n$ and $\sum c_i = 0 \in \mathbb{R}$, then each c_i vanishes.

We will say that K is a *polyhedron* if, for every point $x \in K$, there exists a finite number of linear simplices $\sigma_i \subseteq K$ such that the union $\bigcup_i \sigma_i$ contains a neighborhood of X.

Remark 4. Any open subset of a polyhedron in $\mathbb{R}^n$ is again a polyhedron.

Remark 5. Every polyhedron $K \subseteq \mathbb{R}^n$ admits a *triangulation*: that is, we can find a collection of linear simplices $S = \{\sigma_i \subseteq K\}$ with the following properties:

(1) Any face of a simplex belonging to S also belongs to S.

(2) Any nonempty intersection of any two simplices of S is a face of each.

(3) The union of the simplices σ_i is K.

Definition 6. Let $K \subseteq \mathbb{R}^n$ be a polyhedron. We will say that a map $f : K \to \mathbb{R}^m$ is *linear* if it is the restriction of an affine map from $\mathbb{R}^n$ to $\mathbb{R}^m$. We will say that f is *piecewise linear* (PL) if there exists a triangulation $\{\sigma_i \subseteq K\}$ such that each of the restrictions $f|\sigma_i$ is linear.

If $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^m$ are polyhedra, we say that a map $f : K \to L$ is piecewise linear if the underlying map $f : K \to \mathbb{R}^m$ is piecewise linear.

Remark 7. Let $f : K \to L$ be a piecewise linear homeomorphism between polyhedra. Then the inverse map $f^{-1} : L \to K$ is again piecewise linear. To see this, choose any triangulation of K such that the restriction of f to each simplex of the triangulation is linear. Taking the image under f, we obtain a triangulation of L such that the restriction of f^{-1} to each simplex is linear.

Remark 8. The collection of all polyhedra can be organized into a category, where the morphisms are given by piecewise linear maps. This allows us to think about polyhedra *abstractly*, without reference to an embedding into a Euclidean space: a pair of polyhedra $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^m$ can be isomorphic even if $n \neq m$.

Definition 9. Let M be a polyhedron. We will say that M is a *piecewise linear manifold* (of dimension n) if, for every point $x \in M$, there exists an open neighborhood $U \subseteq M$ containing x and a piecewise linear homeomorphism $U \simeq \mathbb{R}^n$.

Remark 10. Definition 9 can be rephrased so as to better resemble Definition 2. Namely, let M be a topological manifold. We define a *combinatorial structure* on M to be a subsheaf $\mathcal{O}_M^{PL} \subseteq \mathcal{O}_M$ with the following property: for every point $x \in X$, there exists an open embedding $f : \mathbb{R}^n \to M$ whose image contains f, such that $f^* \mathcal{O}_M^{PL} \subseteq \mathcal{O}_{\mathbb{R}^n}$ can be identified with the sheaf whose value on an open subset $U \subseteq \mathbb{R}^n$ consists of piecewise linear maps from U to $\mathbb{R}$.

Every piecewise linear manifold M comes equipped with a combinatorial structure, where we define $\mathcal{O}_M^{PL}$ to be the sheaf of piecewise linear maps on M with values in $\mathbb{R}$. Conversely, if M is a topological manifold endowed with a combinatorial structure, then by choosing sufficiently many sections $f_1, \ldots, f_m \in \mathcal{O}_M^{PL}(M)$ we obtain an embedding $M \to \mathbb{R}^m$ whose image is a polyhedron $\mathbb{R}^m$ (which is a piecewise linear manifold). We can therefore regard the data of a piecewise linear manifold as equivalent to the data of a topological manifold with a combinatorial structure.

Let K be a polyhedron containing a vertex x, and choose a triangulation of K containing x as a vertex of the triangulation. The *star* of x is the union of those simplices of the triangulation which contain x. The *link* of x consists of those simplices belonging to the star of x which do not contain x. We denote the link of x by $\mathrm{lk}(x)$.

As a subset of K, the link $\mathrm{lk}(x)$ of x depends on the choice of triangulation of K. However, one can show that as an abstract polyhedron, $\mathrm{lk}(x)$ is independent of the triangulation up to piecewise linear homeomorphism. Moreover, $\mathrm{lk}(x)$ depends only on a neighborhood of x in K.

If $K = \mathbb{R}^n$ and $x \in K$ is the origin, then the link $\mathrm{lk}(x)$ can be identified with the sphere S^{n-1} (which can be regarded as a polyhedron via the realization $S^{n-1} \simeq \partial \Delta^n$). It follows that if K is any piecewise linear n-manifold, then the link $\mathrm{lk}(x)$ is equivalent to S^{n-1} for every point $x \in K$. Conversely, if K is any polyhedron such that every link in K is an $(n-1)$-sphere, then K is a piecewise linear n-manifold. To see this, we observe that for each $x \in K$, if we choose a triangulation of K containing x as a vertex, then the star of x can be identified with the cone on $\mathrm{lk}(x)$. If $\mathrm{lk}(x) \simeq S^{n-1}$, then the star of x is a piecewise linear (closed) disk, so that x has a neighborhood which admits a piecewise linear homeomorphism to the open disk in $\mathbb{R}^n$.

We have proven the following:

Proposition 11. *Let K be a polyhedron. The following conditions are equivalent:*

(i) For each $x \in K$, the link $\mathrm{lk}(x)$ is a piecewise linear $(n-1)$-sphere.

(ii) K is a piecewise linear n-manifold.

Remark 12. Very roughly speaking, we can think of a piecewise linear manifold M as a topological manifold equipped with a triangulation. However, this is not quite accurate, since a polyhedron does not come equipped with a particular triangulation. Instead, we should think of M as equipped with a distinguished class of triangulations, which is stable under passing to finer and finer subdivisions.

Warning 13. Let K be a polyhedron whose underlying topological space is an n-manifold. Then K need not be a piecewise linear n-manifold: it is generally not possible to choose local charts for K in a piecewise linear fashion.

To get a feel for the sort of problems which might arise, consider the criterion of Proposition 11. To prove that K is a piecewise linear n-manifold, we need to show that for each $x \in K$, the link $\mathrm{lk}(x)$ is a (piecewise-linear) n-sphere. Using the fact that K is a topological manifold, we deduce that $\mathrm{H}_*(K, K - \{x\}; \mathbf{Z})$ is isomorphic to $\mathbf{Z}$ in degree n and zero elsewhere; this is equivalent to the assertion that $\mathrm{lk}(x)$ has the homology of an $(n-1)$-sphere. Of course, this does not imply that $\mathrm{lk}(x)$ is itself a sphere. A famous counterexample is due to Poincare: if we let I denote the binary icosahedral group, regarded as a subgroup of $\mathrm{SU}(2) \simeq S^3$, then the quotient $P = \mathrm{SU}(2)/I$ is a homology sphere which is not a sphere (since it is not simply connected).

The suspension ΣP is a 4-dimensional polyhedron whose link is isomorphic to P at precisely two points, which we will denote by x and y. However, ΣP is not a topological manifold. To see this, we note that the point x does not contain arbitrarily small neighborhoods U such that $U - \{x\}$ is simply connected. In other words, the failure of ΣP to be a manifold can be detected by computing the *local fundamental group of* $P - \{x\}$ *near* x (which turns out to be isomorphic to the fundamental group of P). However, if we apply the suspension functor again, the same considerations do not apply: the space ΣP is simply connected (by van Kampen's theorem). Surprisingly enough, it turns out to be a manifold:

Theorem 14 (Cannon-Edwards)**.** *Let P be a topological n-manifold which is a homology sphere. Then the double suspension $\Sigma^2 P$ is homeomorphic to an $(n+2)$-sphere.*

In particular, if we take P to be the Poincare homology sphere, then there is a homeomorphism $\Sigma^2 P \simeq S^5$. However, $\Sigma^2 P$ is not a piecewise linear manifold: it contains two points whose links are given by ΣP, which is not even a topological 4-manifold (let alone a piecewise linear 4-sphere).

The upshot of Warning 13 is that a topological manifold M (such as the 5-sphere) admits triangulations which are badly behaved, in the sense that the underlying polyhedron is not locally equivalent to Euclidean space. The situation is different if we require our triangulations to be compatible with a smooth structure on M. We will take this point up in the next lecture.