Product Structure Theorem: First Steps (Lecture 19)

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In the last lecture, we saw that the connectivity properties of the map $PL(m)/O(m) \to PL(m+1)/O(m+1)$ could be phrased geometrically as follows:

Theorem 1 (Product Structure Theorems)**.** *Let M be a PL manifold of dimension m, let $K \subseteq M$ be a closed subpolyhedron, and suppose we are given a smooth structure on $M \times \mathbb{R}$ which is the product of a smooth structure on M with the standard smooth structure on $\mathbb{R}$ in a neighborhood of $K \times \mathbb{R}$. Then, after modifying the smooth structure by a suitable PD isotopy which is trivial in a neighborhood of $K \times \mathbb{R}$, we can arrange that the smooth structure on $M \times \mathbb{R}$ is the product of a smooth structure on M with the standard smooth structure on $\mathbb{R}$. The same result holds if we replace $\mathbb{R}$ by $[0,1]$.*

Our goal in the next few lectures is to sketch a proof of this result. The argument is essentially the same whether we use $\mathbb{R}$ or $[0,1]$; we will therefore switch from one case to the other as convenient. To simplify the exposition, we will assume that $K = \emptyset$. The case where K is nonempty can be treated by more careful versions of the same arguments.

To begin, let us assume that we are given a smooth structure on the product $M \times [0,1]$. Let $X = M \times [0,1]$, and let $\pi : X \to [0,1]$ denote the projection. The easiest case of Theorem 1 is the following:

Lemma 2. *Theorem 1 is true if π is a smooth submersion.*

Proof. If π is a smooth submersion, then it exhibits X as a smooth fiber bundle over $[0,1]$. Let $M_0 = M$, equipped with the smooth structure given by the identification $M_0 \simeq \pi^{-1}\{0\}$. We have a diffeomorphism $f : X \simeq M_0 \times [0,1]$.. In other words, X is diffeomorphic to a product with $[0,1]$. This is not quite the full strength of Theorem 1: we must show that this diffeomorphism can be chosen to be PD isotopic to the identity map on X. Let us think of f as a PD family $\{f_t : M \to M_0\}_{t \in [0,1]}$ of PD homeomorphisms from M to M_0, where f_0 is the identity. Define a PD isotopy $\{h_t : X \to M_0 \times [0,1]\}_{t \in [0,1]}$ by the formula

$$h_t(m,s) = \begin{cases} (f_{s-t}(m), s) & \text{if } t \leq s \\ (f_0(m), s) & \text{if } t \geq s. \end{cases}$$

Then h_0 is the diffeomorphism f, which gives the original smooth structure on X. The map h_1 is the identity map $X \simeq M \times [0,1] \simeq M_0 \times [0,1]$, which gives a product smooth structure on X. □

If π is a smooth map, then we can test whether or not π is a submersion by checking whether the derivative of π does not vanish at any point. Of course, the condition that π is smooth is very strong: in our situation, we only know that π is piecewise linear with respect to some Whitehead compatible triangulation of X. In other words, we know that π is piecewise differentiable on X: that is, there is a smooth triangulation of X such that π is differentiable on each simplex. In this case, it is still possible to salvage something of the theory of derivatives:

Definition 3. Let X be a smooth manifold, and let $f : X \to \mathbb{R}$ be a piecewise differentiable map. (In the case of interest, X is a smoothing of $M \times \mathbb{R}$ for some PL manifold M, and f is the projection onto the second factor.) Let $x \in X$ be a point and let v be a tangent vector to X and x. We define $D_v(f)$ to be the minimum value of the derivatives $D_v(f|\sigma)$, where σ ranges over all simplices containing x of some triangulation of X for which f is smooth on each simplex.

The map $(v, x) \mapsto D_v(f)$ is not generally continuous if f is not a smooth function. However, it is lower semicontinuous. In other words, for every real number ϵ, the subset of the tangent bundle T_X consisting of pairs (x, v) for which $D_v(f) > \epsilon$ is an open set. We will say that a tangent vector v to X is *regular for f* if $D_v(f) > 0$. Lower semicontinuity guarantees that the set of regular tangent vectors is open in T_X.

Definition 4. Let X be a smooth manifold and $f : X \to \mathbb{R}$ a piecewise differentiable function. We will say that f is *regular* if, for every point $x \in X$, there exists a tangent vector $v \in T_{X,x}$ such that (x, v) is regular (in other words, such that $D_v(f) > 0$).

Example 5. If f is smooth, then f is regular if and only if it is a smooth submersion.

Lemma 6. *Let X be a smooth manifold and $f : X \to \mathbb{R}$ a regular piecewise differentiable function. Then there exists a smooth tangent field $v : X \to T_X$ such that, for every $x \in X$, the tangent vector $v(x)$ is regular for f.*

Proof. Since f is regular, we can find for each x a tangent vector w_x at x such that $D_{v_x}(f) > 0$. Let $v_x : X \to T_X$ be a smooth tangent field such that $v_x(x) = w_x$. Since the collection of regular tangent vectors is open, there exists an open neighborhood U_x of x such that $v_x(y)$ is f-regular for $y \in U_x$. Since X is paracompact, the open covering $\{U_x\}_{x \in X}$ has a locally finite refinement. Choose a smooth partition of unity ψ_i subordinate to this refinement, so that each ψ_i is supported in U_{x_i}. Then the smooth vector field $v = \sum_i \psi_i v_{x_i}$ has the desired property. □

In the situation of Lemma 6, we will say that the vector field f is *transverse* to f.

Lemma 7. *Let $f : X \to \mathbb{R}$ be a piecewise differentiable function, and let $v : X \to T_X$ be a smooth vector field which is transverse to f. Then for any continuous function $\epsilon : X \to \mathbb{R}_{>0}$, there exists a smooth map $g : X \to \mathbb{R}$ such that*

$$D_{v(x)}(g) > D_{v(x)}(f) - \epsilon(x)$$

$$g(x) - f(x) < \epsilon(x).$$

(Choosing ϵ sufficiently small will guarantee that v is also transverse to g.)

Proof. Choose a partition of unity ψ_i on X subordinate to a locally finite cover of X by compact sets K_i, each of which is contained in a coordinate chart U_i. Suppose we are given smooth maps $g_i : U_i \to \mathbb{R}$, and define g by the formula

$$g = \sum \psi_i g_i.$$

Then $g(x) - f(x) < \epsilon(x)$ will be satisfied provided that $g_i(x) - f(x) < \epsilon(x)$ holds for $x \in U_i$. The other condition is a bit more subtle: we have

$$\begin{aligned} D_{v(x)} g &= \sum_i (D_{v(x)} \psi_i) g_i + \sum_i \psi_i D_{v(x)}(g_i) \\ &= \sum_i (D_{v(x)} \psi_i)(g_i - f) + D_{v_x}(\sum_i \psi_i) f + \sum_i \psi_i D_{v(x)}(g_i) \\ &\geq \sum_i \psi_i D_{v(x)}(g_i) - \sum_i C_i (g_i - f) \end{aligned}$$

where $C_i > 0$ is an upper bound for the compactly supported function $D_{v(x)} \psi_i$. If the inequalities

$$D_{v(x)}(g_i) > D_{v(x)}(f) - \frac{\epsilon(x)}{2}$$

$$\sum_{x \in K_j \cap K_i} C_j (g_j(x) - f(x)) < \frac{\epsilon(x)}{2}$$

hold for $x \in K_i$, then g will satisfy the desired inequality. Since only finitely many intersections $K_j \cap K_i$ are nonempty, the latter inequality can be achieved by ensuring that each g_i is a close approximation to f on K_i.

In other words, we may reduce to the case where $X = \mathbb{R}^n$, and the inequalities

$$D_{v(x)}(g) > D_{v(x)}(f) - \epsilon(x)$$

$$g(x) - f(x) < \epsilon(x).$$

only need to be satisfied when x lies in some compact subset $K \subseteq \mathbb{R}^n$. Let $k : \mathbb{R}^n \to \mathbb{R}_{>0}$ be a smooth function with total integral 1, which is supported in a small ball of radius δ. Define $g(x) = \int_y f(y)k(x-y)$. Then g is a smooth function. It is not difficult to see that the conditions

$$D_{v(x)}(g) > D_{v(x)}(f) - \epsilon(x)$$

$$g(x) - f(x) < \epsilon(x).$$

will be satisfied on any compact subset K, provided that δ is chosen sufficiently small. □

We now come to the main goal of this lecture:

Proposition 8. *Theorem 1 is true in the case where the projection $\pi : M \times \mathbb{R} \to \mathbb{R}$ is a regular (but not necessarily smooth with respect the smoothing of $M \times \mathbb{R}$).*

Proof. We will show that, after adjusting the smooth structure on $M \times \mathbb{R}$ by a PD isotopy, we can arrange that π is a smooth submersion; the desired result will then follow from Lemma 2. First, choose a smooth Riemannian metric on $X = M \times \mathbb{R}$. Let $\epsilon : X \to \mathbb{R}_{>0}$ be a smooth function such that each of the closed balls $B_{\epsilon(x)}(x) \subseteq X$ of radius $\epsilon(x)$ around x is compact. Let $v : X \to T_X$ be a smooth tangent field which is transverse to π. Rescaling v, we can assume that each $v(x)$ has unit length.

Choose a smooth function $\delta : X \to \mathbb{R}_{>0}$ such that

$$D_{v(x)}(f) > \delta(x)$$

for $x \in X$. Let $\delta' : X \to \mathbb{R}_{>0}$ be another smooth function such that if $d(x,y) \leq \epsilon$, then $\delta'(x) \leq \delta(y)$. Using the previous Lemma, we can choose a smooth map $g : X \to \mathbb{R}$ with the following properties:

$$D_{v(x)}(g) > \frac{\delta(x)}{2}$$

$$\pi(x) - g(x) < \epsilon(x)\frac{\delta'(x)}{2}.$$

In particular, $\lambda(x) = D_{v(x)}(g)$ is a smooth function of x satisfying $\pi(x) - g(x) < \epsilon(x)\lambda(y)$ whenever $d(x,y) < \epsilon(x)$.

Since v is a unit vector field and each of the $\epsilon(x)$-balls around x is compact, the flow along the vector field v gives a well-defined map

$$F : \{(x,t) \in X \times \mathbb{R} : |t| < \epsilon(x)\} \to X.$$

Moreover, for fixed x, $F(x,t)$ stays in a ball of radius ϵ around x. It follows that the t-derivative of $g(F(x,t))$ coincides with $\lambda(F(x,t)) > \frac{f(x)-g(x)}{\epsilon(x)}$. Consequently, for $s \in [0,1]$, we can find a unique $t = t(x,s)$ such that $g(F(x,t)) - g(x) = s(\pi(x) - g(x))$. We now define a map $h_s : X \to X$ by the formula

$$h_s(x) = F(x, t(x,s)).$$

The family $\{h_s : X \to X\}_{s \in [0,1]}$ is then a PD isotopy from X to itself, where h_0 is the identity and $g \circ h_1 = f$, so that f is smooth with respect to the smooth structure on X determined by h_1. □