# Product Structure Theorems (Lecture 18)

March 16, 2009

Our goal in this lecture is to study the relative connectivity properties of the quotient spaces $PL(m)/O(m)$. Our basic observation is the following:

**Remark 1.** Let $K \subseteq \mathbb{R}^m$ be a closed subpolyhedron. Then the mapping space $(PL(m)/O(m))^K$ can be identified with the simplicial set $\mathrm{Smooth}(K)$ of germs of smooth structures on $\mathbb{R}^m$ near $K$. This follows from the main result of the last lecture, together with the observation that the standard PL structure on $\mathbb{R}^m$ determines a *constant* map $\chi : \mathbb{R}^m \to BPL(m)$.

**Proposition 2.** *Fix an integer $m \geq 0$. The following conditions are equivalent:*

(1) *All homotopy fibers of the map $PL(m)/O(m) \to PL(m+1)/O(m+1)$ are $(m-1)$-connected.*

(2) *All homotopy fibers of the map $BO(m) \to BO(m+1) \times_{BPL(m+1)} BPL(m)$ are $(m-1)$-connected.*

(3) *The following weak product structure theorem holds:*

  $(*)$ *Let $M$ be a PL manifold of dimension $m$, let $K \subseteq M$ be a closed subpolyhedron, and suppose we are given a smooth structure on $M \times \mathbb{R}$ which is the product of a smooth structure on $M$ with the standard smooth structure on $\mathbb{R}$ in a neighborhood of $K \times \mathbb{R}$. Then, after modifying the smooth structure by a suitable PD isotopy which is trivial in a neighborhood of $K \times \mathbb{R}$, we can arrange that the smooth structure on $M \times \mathbb{R}$ is the product of a smooth structure on $M$ with the standard smooth structure on $\mathbb{R}$.*

*Proof.* We have a natural transformation of homotopy fiber sequences

$$\begin{array}{ccccc} PL(m)/O(m) & \longrightarrow & BO(m) & \longrightarrow & BPL(m) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \downarrow \\ PL(m+1)/O(m+1) & \xrightarrow{\theta} & BO(m+1) \times_{BPL(m+1)} BPL(m) & \longrightarrow & BPL(m). \end{array}$$

It follows that every homotopy fiber of $\phi$ is also a homotopy fiber of $\psi$, so the implication (2) $\Rightarrow$ (1) is clear. To prove the converse, it suffices to show that every homotopy fiber of $\psi$ is equivalent to a homotopy fiber of $\phi$. This will follow if the map $\theta$ is surjective on $\pi_0$. This surjectivity follows from the fiber sequence, since $BPL(m)$ is connected.

We now prove that (2) $\Rightarrow$ (3). In the situation of (3), the smooth structure on $M \times \mathbb{R}$ is classified by a map $M \times \mathbb{R} \to BO(m+1) \times_{PL(m+1)} PL(m)$. Finding a PD isotopy to a smooth structure on $M \times \mathbb{R}$ which is a product with $\mathbb{R}$ is equivalent to solving the lifting problem

$$\begin{array}{ccc} & & BO(m) \\ & \nearrow & \downarrow \\ M \times \mathbb{R} & \longrightarrow & BO(m) \times_{BPL(m)} BPL(m). \end{array}$$

If we wish to do achieve this via an isotopy fixed near $K$, then we must solve instead a relative lifting problem of the form

$$\begin{array}{ccc} K \times \mathbb{R} & \longrightarrow & BO(m) \\ \downarrow & \nearrow & \downarrow j \\ M \times \mathbb{R} & \longrightarrow & BO(m+1) \times_{BPL(m+1)} BPL(m). \end{array}$$

This is a purely homotopy theoretic problem; we may therefore replace the inclusion $K \times \mathbb{R} \subseteq M \times \mathbb{R}$ by $K \subseteq M$. Since $M$ is a PL $m$-manifold, it can be obtained from $K$ by successive cell attachments where the cells have dimension $\leq m$. Working cell-by-cell, we are reduced to solving lifting problems of the form

$$\begin{array}{ccc} \partial D^k & \longrightarrow & BO(m) \\ \downarrow & \nearrow & \downarrow \psi \\ D^k & \longrightarrow & BO(m+1) \times_{BPL(m+1)} BPL(m) \end{array}$$

where $D^k$ indicates a disk of dimension $\leq k$. The obstruction to solving such a problem is equivalent to the vanishing of a class in $\pi_{k-1}$ of a homotopy fiber $F$ of $\psi$. This class automatically vanishes by virtue of our assumption that $F$ is $(m-1)$-connected.

We now prove that (3) $\Rightarrow$ (1). We must show that every lifting problem of the form

$$\begin{array}{ccc} \partial D^k & \longrightarrow & PL(m)/O(m) \\ \downarrow & \nearrow & \downarrow \psi \\ D^k & \longrightarrow & PL(m+1)/O(m+1) \end{array}$$

has a solution, provided that $k \leq m$. In this case, we can choose a PL embedding of $\partial D^k$ into $\mathbb{R}^m$ and obtain an equivalent lifting problem

$$\begin{array}{ccc} \partial D^k \times \mathbb{R} & \longrightarrow & PL(m)/O(m) \\ \downarrow & \nearrow & \downarrow \psi \\ \mathbb{R}^{m+1} & \longrightarrow & PL(m+1)/O(m+1). \end{array}$$

The diagram determines a smoothing of $\mathbb{R}^{m+1}$ which is a product smoothing in a neighborhood of $\partial D^k \times \mathbb{R}$, and a solution to the indicated lifting problem is equivalent to giving a PD isotopy (fixed near $\partial D^k \times \mathbb{R}$) to a product smoothing. □

**Remark 3.** If the equivalent conditions of Proposition 2 are satisfied, then the map $PL(m)/O(m) \to PL(m+1)/O(m+1)$ is surjective on $\pi_0$ for $m \geq 0$. Since $PL(0)/O(0) = *$ is connected, we it follows by induction that $PL(m)/O(m)$ is connected for each $m$. In other words, Euclidean space $\mathbb{R}^m$ admits a unique smooth structure compatible with its standard PL structure, up to PD isotopy.

The connectivity estimate given in Proposition 2 is not the best possible. We now describe how to do a little better. We need a variation on the main result of the last lecture, which applies to manifolds with boundary.

**Variant 4.** *Let $M$ be a PL $(m+1)$-manifold with boundary $\partial M$. We can define the notion of a smoothing of $M$ as before. Smoothings of $M$ can be organized into a simplicial set* $\mathrm{Smooth}(M)$*. Every smoothing of $M$ determines a smoothing of the boundary of $M$; this is given by a Kan fibration* $\mathrm{Smooth}(M) \to \mathrm{Smooth}(\partial M)$*. Given a smooth structure on the boundary of $M$, we denote the fiber of this map by* $\mathrm{Smooth}(M; \partial)$*. Given* *such a smoothing of $\partial M$, we get a map $\partial M \to BO(m)$. Then, up to homotopy, smoothings of $M$ compatible with this smooth structure on $\partial M$ are given by solutions to the lifting problem*

$$\begin{array}{ccc} \partial M & \longrightarrow & BO(m+1) \\ \downarrow & \nearrow & \downarrow \\ M & \longrightarrow & BPL(m+1). \end{array}$$

*Smoothings of $M$ itself (without boundary data) can be identified with solutions to the lifting problem of pairs*

$$\begin{array}{ccc} & & (BO(m+1), BO(m)) \\ & \nearrow & \downarrow \\ (M, \partial M) & \longrightarrow & (BPL(m+1), BPL(m)). \end{array}$$

**Notation 5.** Fix an integer $m \geq 0$. We let $\Delta_m^{PL}$ denote the homotopy fiber product

$$BPL(m) \times^h_{BPL(m+1)} BPL(m) = BPL(m) \times_{BPL(m+1)^{\{0\}}} BPL(m+1)^{[0,1]} \times_{BPL(m+1)^{\{1\}}} BPL(m).$$

Similarly, define $\Delta_m^O$ to be the homotopy fiber product

$$BO(m) \times^h_{BO(m+1)} BO(m) = BO(m) \times_{BO(m+1)^{\{0\}}} BO(m+1)^{[0,1]} \times_{BO(m+1)^{\{1\}}} BO(m).$$

We have a Kan fibration $\Delta^O \to \Delta^{PL}$. For every PL $m$-manifold $M$, the tangent microbundle to $M \times [0,1]$ and its boundary determines a map $M \to \Delta_m^{PL}$. According to Variation 4, we can identify smoothings of $M \times [0,1]$ with solutions to the lifting problem

$$\begin{array}{ccc} & & \Delta_m^O \\ & \nearrow & \downarrow \\ M & \longrightarrow & \Delta_m^{PL}. \end{array}$$

The proof of Proposition 2 adapts without essential change to show the following:

**Proposition 6.** *Fix an integer $m \geq 0$. The following conditions are equivalent:*

(1) *Let $F$ denote the homotopy fiber of the map $\Delta_m^O \to \Delta_m^{PL}$. Then all $PL(m)/O(m) \to F$ are $(m-1)$-connected.*

(2) *All homotopy fibers of the map $BO(m) \to \Delta_m^O \times_{\Delta_m^{PL}} BPL(m)$ are $(m-1)$-connected.*

(3) *The following strong product structure theorem holds:*

   $(*)$ *Let $M$ be a PL manifold of dimension $m$, let $K \subseteq M$ be a closed subpolyhedron, and suppose we are given a smooth structure on $M \times [0,1]$ which is the product of a smooth structure on $M$ with the standard smooth structure on $[0,1]$ in a neighborhood of $K \times \mathbb{R}$. Then, after modifying the smooth structure by a suitable PD isotopy which is trivial in a neighborhood of $K \times [0,1]$, we can arrange that the smooth structure on $M \times [0,1]$ is the product of a smooth structure on $M$ with the standard smooth structure on $[0,1]$*

**Remark 7.** Let $F$ be as in Proposition 6. Then the homotopy fibers of the map $PL(m)/O(m) \to F$ can be identified with *path spaces* in the space in homotopy fibers of the map $\psi : PL(m)/O(m) \to PL(m+1)/O(m+1)$. Consequently, if we grant that the homotopy fibers of $\psi$ are nonempty (which follows from

Proposition 2 if $m \geq 0$), then Proposition 6 asserts that the homotopy fibers of $\psi$ are $m$-connected. This is a slightly better connectivity estimate than we get from Proposition 2 itself, which is why the geometric assertion of part (3) of Proposition 6 is called the *strong* product structure theorem to contrast it with the corresponding *weak* product structure theorem of Proposition 2. However, the terminology is slightly misleading: Proposition 6 does not quite formally imply Proposition 2, since it does not guarantee that the homotopy fibers of $\psi$ are nonempty. This missing strength is equivalent to the assertion of Remark 3: we need to know that every smooth structure on $\mathbb{R}^m$ is PD isotopic to the product with $\mathbb{R}$ of a smooth structure on $\mathbb{R}^{m-1}$, and thus (using induction on $m$) PD isotopic to the standard smooth structure on $\mathbb{R}^m$.