Introduction (Lecture 1)

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One of the basic problems of manifold topology is to give a classification for manifolds (of some fixed dimension n) up to diffeomorphism. In the best of all possible worlds, a solution to this problem would provide the following:

(i) A list of n-manifolds $\{M_\alpha\}$, containing one representative from each diffeomorphism class.

(ii) A procedure which determines, for each n-manifold M, the unique index α such that $M \simeq M_\alpha$.

In the case $n = 2$, it is possible to address these problems completely: a connected oriented surface Σ is classified up to homeomorphism by a single integer g, called the *genus* of Σ. For each $g \geq 0$, there is precisely one connected surface Σ_g of genus g up to diffeomorphism, which provides a solution to (i). Given an arbitrary connected oriented surface Σ, we can determine its genus simply by computing its Euler characteristic $\chi(\Sigma)$, which is given by the formula $\chi(\Sigma) = 2 - 2g$: this provides the procedure required by (ii).

Given a solution to the classification problem satisfying the demands of (i) and (ii), we can extract an algorithm for determining whether two n-manifolds M and N are diffeomorphic. Namely, we apply the procedure (ii) to extract indices α and β such that $M \simeq M_\alpha$ and $N \simeq M_\beta$: then $M \simeq N$ if and only if $\alpha = \beta$. For example, suppose that $n = 2$ and that M and N are connected oriented surfaces with the same Euler characteristic. Then the classification of surfaces tells us that there is a diffeomorphism ϕ from M to N. In practice, we might want to apply this information by using ϕ to make some other construction. In this case, it is important to observe that ϕ is not unique: there are generally many different diffeomorphisms from M to N.

Example 1. Let M be a compact oriented 3-manifold, and suppose we are given a submersion $M \to S^1$. Fix a base point $* \in S^1$. The fiber $M \times_{S^1} *$ is a compact oriented surface which we will denote by Σ. Write S^1 as a quotient $[0,1]/\{0,1\}$, so that M is obtained from the pullback $M' = M \times_{S^1} [0,1]$ by gluing together the fibers $M'_0 \simeq \Sigma$ and M'_1. Since the interval $[0,1]$ is contractible, we can write M' as a product $\Sigma \times [0,1]$. In order to recover M from M', we need to supply a diffeomorphism of $M' \simeq \Sigma$ to $M'_1 \simeq \Sigma$: in other words, we need to supply a diffeomorphism ϕ of Σ with itself. The diffeomorphism ϕ depends on a choice of identification $M' \simeq \Sigma \times [0,1]$. If we assume that this diffeomorphism is normalized to be the identity on M'_0, then we see that ϕ is well-defined up to *isotopy* (recall that two diffeomorphisms $\gamma_0, \gamma_1 : \Sigma \to \Sigma$ are *isotopic* if there is a continuous family $\{\gamma_t : \Sigma \to \Sigma\}_{t \in [0,1]}$ of diffeomorphisms which interpolates between γ_0 and γ_1).

Motivated by this example, it is natural to refine our original classification problem: given two n-manifolds M and N, we would like to know not only whether M and N are diffeomorphic, but to have a classification of all diffeomorphisms from M to N, at least up to isotopy. Note that the collection $\mathrm{Diff}(M,N)$ of diffeomorphisms from M to N carries a natural topology, and the isotopy classes of diffeomorphisms from M to N can be identified with elements of the set $\pi_0 \mathrm{Diff}(M,N)$ of path components of $\mathrm{Diff}(M,N)$. Our goal in this class is to address the following more refined question:

Problem 2. *Given a pair of n-manifolds M and N, determine the homotopy type of the space* $\mathrm{Diff}(M,N)$.

Remark 3. The space $\mathrm{Diff}(M,N)$ is nonempty if and only if M and N are diffeomorphic. If $\mathrm{Diff}(M,N)$ is nonempty, then $M \simeq N$ so we can identify $\mathrm{Diff}(M,N)$ with the group $\mathrm{Diff}(M) = \mathrm{Diff}(M,M)$ of diffeomorphisms from M to itself.

Remark 4. One might ask why Problem 2 is addressing the right question. For example, why do we want to understand the homotopy type of $\mathrm{Diff}(M,N)$ as opposed to some more precise invariant (like the topological space $\mathrm{Diff}(M,N)$ itself) or less precise invariant (like the set $\pi_0 \mathrm{Diff}(M,N)$)?

One answer is that the exact topological space $\mathrm{Diff}(M,N)$ depends on exactly what we mean by a diffeomorphism. For example, should we work with diffeomorphisms that are merely differentiable, or should they be infinitely differentiable? The exact topological space $\mathrm{Diff}(M,N)$ will depend on how we answer this question. But, as we will see later, the homotopy type of $\mathrm{Diff}(M,N)$ does not.

To motivate why we would like to understand the entire homotopy type of $\mathrm{Diff}(M,N)$, rather than just its set of path components, we remark that Example 1 can be generalized as follows: given a pair of compact manifolds M and B, the collection of isomorphism classes of smooth fiber bundles $E \to B$ with fiber M can be identified with the collection of homotopy classes of maps from B into the classifying space $B\,\mathrm{Diff}(M)$. In other words, understanding the homotopy types of the groups $\mathrm{Diff}(M)$ is equivalent to understanding the classification of *families* of manifolds.

Problem 2 is very difficult in general. To address it, it is useful to divide manifolds into two different "regimes":

- If $n \geq 5$, then we are in the world of high-dimensional topology. In this case, it is possible to obtain partial information about the homotopy type of $\mathrm{Diff}(M)$ (for example, a description of its rational homotopy groups in a range of degrees) using the techniques of surgery theory. The techniques for obtaining this information are generally algebraic in nature (involving Waldhausen K-theory and L-theory).
- If $n \leq 4$, then we are in the world of low-dimensional topology. In this case, it is customary to approach Problem 2 using geometric and combinatorial techniques. The success of these method is highly dependent on n.

Our goal in this course is to study Problem 2 in the low-dimensional regime. When $n = 4$, very little is known about Problem 2: for example, little is known about the homotopy type of the diffeomorphism group $\mathrm{Diff}(S^4)$. We will therefore restrict our attention to manifolds of dimension n for $1 \leq n \leq 3$. We will begin in this lecture by studying the case $n = 1$. In this case, there is only one connected closed 1-manifold up to diffeomorphism: the circle S^1. However, we can study S^1 from many different points of view:

- Geometry: We can regard the circle S^1 as a Riemannian manifold, and study its isometry group $\mathrm{Isom}(S^1)$.
- Differential topology: We can regard the circle S^1 as a smooth manifold, and study its diffeomorphism group $\mathrm{Diff}(S^1)$.
- Point-Set Topology: We can regard the circle S^1 as a topological manifold, and study the group $\mathrm{Homeo}(S^1)$ of homeomorphisms of S^1 with itself.
- Homotopy Theory: We can ignore the actual topology of S^1 in favor of its homotopy type, and study the monoid $\mathrm{Self}(S^1)$ of homotopy equivalences $S^1 \to S^1$.

We have evident inclusions

$$\mathrm{Isom}(S^1) \subseteq \mathrm{Diff}(S^1) \subseteq \mathrm{Homeo}(S^1) \subseteq \mathrm{Self}(S^1).$$

Theorem 5. *Each of the above inclusions is a homotopy equivalence.*

Proof. Each of the spaces above can be decomposed into two pieces, depending on whether or not the underlying map preserves or reverses orientations. Consider the induced sequence

$$\mathrm{Isom}^+(S^1) \subseteq \mathrm{Diff}^+(S^1) \subseteq \mathrm{Homeo}^+(S^1) \subseteq \mathrm{Self}^+(S^1)$$

where the superscript indicates that we restrict our attention to orientation-preserving maps. The group $\mathrm{Isom}^+(S^1)$ is homeomorphic to the circle S^1 itself: an orientation-preserving isometry from S^1 to itself is just given by a rotation. The other groups admit decompositions

$$\mathrm{Diff}^+(S^1) = \mathrm{Diff}^+_0(S^1)\,\mathrm{Isom}^+(S^1)$$

$$\mathrm{Homeo}^+(S^1) = \mathrm{Homeo}^+_0(S^1)\,\mathrm{Isom}^+(S^1)$$

$$\mathrm{Self}^+(S^1) = \mathrm{Self}^+_0(S^1)\,\mathrm{Isom}^+(S^1),$$

where the subscript 0 indicates that we consider maps from S^1 to itself which fix a base point $* \in S^1$. To complete the proof, it will suffice to show that the spaces $\mathrm{Diff}^+_0(S^1)$, $\mathrm{Homeo}^+_0(S^1)$, and $\mathrm{Self}^+_0(S^1)$ are contractible.

We first treat the case of $\mathrm{Self}^+_0(S^1)$. We note that the circle S^1 can be identified with the quotient $\mathbb{R}/\mathbf{Z}$. If f is a map from the circle S^1 to itself which preserves the base point (the image of $0 \in \mathbb{R}$), then we can lift f to a base-point preserving map $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ satisfying $\widetilde{f}(x+1) = \widetilde{f}(x) + d$, where d is the degree of the map $f : S^1 \to S^1$. Conversely, any map $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ satisfying this condition descends to give a map $f : S^1 \to S^1$ of degree d. We observe that f is a homotopy equivalence if and only if $d = \pm 1$, and that f is an orientation-preserving homotopy equivalence if and only if $d = 1$. We may therefore identify $\mathrm{Self}^+_0(S^1)$ with the space

$$V = \{\widetilde{f} : \mathbb{R} \to \mathbb{R} : \widetilde{f}(0) = 0 \wedge \widetilde{f}(x+1) = \widetilde{f}(x) + 1\}$$

We wish to prove that V is contractible. In fact, for any element $\widetilde{f} \in V$, there is a canonical path from $\widetilde{f} = \widetilde{f}_0$ to the identity map $\mathrm{id}_{\mathbb{R}} = \widetilde{f}_1$, given by the formula

$$\widetilde{f}_t(x) = (1-t)\widetilde{f}(x) + tx.$$

We can use the identification $\mathrm{Self}^+_0(S^1) \simeq V$ to identify $\mathrm{Homeo}^+_0(S^1)$ and $\mathrm{Diff}^+_0(S^1)$ with subsets of V: the former can be identified with the collection of all strictly increasing functions $\widetilde{f} \in V$, and the latter with the collection of all maps $\widetilde{f} \in V$ which are smooth and have nowhere vanishing derivative. Exactly the same contracting homotopy shows that these spaces are contractible as well. □

We can summarize Theorem 5 as follows:

(1) There is essentially no difference between smooth 1-manifolds and topological 1-manifolds.

(2) Every smooth 1-manifold M admits a Riemannian metric which accurately reflects its topology, in the sense that every diffeomorphism of M can be canonically deformed to an isometry.

(3) A 1-manifold M is determined, up to canonical homeomorphism, by its homotopy type.

In this course, we will study to what extent these assertions can be generalized to manifolds of dimensions 2 and 3. Here is a loose outline of the material we might cover in this class:

- In large dimensions, there is an appreciable difference between working with smooth and topological manifolds. A famous example is Milnor's discovery that there exist nondiffeomorphic smooth structures on the sphere S^7. In fact, these differences are apparent already in lower dimensions: Milnor's example comes from the fact that there exist diffeomorphisms of the standard sphere S^6 which are topologically isotopic but not smoothly isotopic, and a similarly the inclusion $\mathrm{Diff}(S^5) \to \mathrm{Homeo}(S^5)$ fails to be a homotopy equivalence. Even more dramatic failures occur in dimension 4: the topological space $\mathbb{R}^4$ can be endowed with uncountably many nondiffeomorphic smooth structures. However, in dimensions

≤ 3 these difficulties do not occur. Namely, one can show that the classification of manifolds (including information about the homotopy types of automorphism groups) of dimension ≤ 3 is the same in the smooth, topological, and piecewise linear categories. The first part of this course will be devoted to making this statement more precise and sketching how it can be proved.

- If Σ is a closed oriented surface of genus $g > 0$, then Σ is aspherical: the homotopy groups $\pi_i(\Sigma)$ vanish for $i > 1$. It follows that the homotopy type of Σ is determined by its fundamental group. In this case, we will also see that the diffeomorphism group $\mathrm{Diff}(\Sigma)$ is homotopy equivalent to the monoid of self-diffeomorphisms $\mathrm{Self}(\Sigma)$, so that $\mathrm{Diff}(\Sigma)$ can be described in an entirely combinatorial way in terms of the fundamental group $\pi_1\Sigma$.

  For 3-manifolds, the situation is a bit more complicated. A general 3-manifold M need not be aspherical: the group $\pi_2(M)$ usually does not vanish. However, via somewhat elaborate geometric arguments one can use the nonvanishing of $\pi_2(M)$ to construct embedded spheres in M which cut M into aspherical pieces (except in a few exceptional cases). The homotopy type of an aspherical manifold M is again determined by the fundamental group $\pi_1(M)$. In many cases, one can show that M is determined up to diffeomorphism by $\pi_1(M)$: this is true whenever M is a *Haken manifold*. We will study the theory of Haken manifolds near the end of this course. (Another case in which M can be recovered from the fundamental group $\pi_1 M$ occurs when M is a hyperbolic 3-manifold: this is the content of *Mostow's rigidity theorem*.)

- Manifolds of dimension 2 and 3 can be fruitfully studied by endowing them with additional structure. For example, we can gain a lot of information about surfaces by choosing conformal structures and then applying the methods of complex analysis. Using the uniformization theorem, one can show that every 2-manifold admits a Riemannian metric of constant curvature: this curvature is positive for the case of a 2-sphere, zero for a torus, and otherwise negative. In dimension 3, Thurston's geometrization conjecture provides a much more complicated but somewhat analogous picture: every 3-manifold can be broken into pieces which admit "geometric structures". If time allows, we will discuss this near the end of the course.