Proof of the Kan Property (Lecture 9)

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Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$. Let B be the polarization of Q and $\mathbb{D}$ the associated duality functor. Our goal in this lecture is to prove the following:

Theorem 1. *Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a stable subcategory which is closed under $\mathbb{D}$. Then the canonical map $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet \to \mathrm{Poinc}(\mathcal{C} / \mathcal{C}_0, Q'')_\bullet$ is a Kan fibration of simplicial spaces (here Q'' is the quadratic functor on $\mathcal{C} / \mathcal{C}_0$ defined in the previous lecture).*

Before giving the proof, we embark on two digressions.

Notation 2. We let $\mathrm{Lagr}(\mathcal{C}, Q)$ denote the full subcategory of $\mathcal{C}_{[1]}$ spanned by those diagrams

$$X_0 \leftarrow X_{01} \to X_1$$

where $X_1 = 0$, and endow $\mathrm{Lagr}(\mathcal{C}, Q)$ with the quadratic functor Q_{Lagr} given by the restriction of $Q_{[1]}$.

Lemma 3. *We have $L_0(\mathrm{Lagr}(\mathcal{C}, Q), Q_{\mathrm{Lagr}}) = 0$.*

Remark 4. Lemma 3 can be regarded as an algebraic analogue of the following topological fact: for any topological space X with a base point $x \in X$, the space of paths in X ending in x is contractible.

Proof. Fix a Poincare object (X, q) of $\mathrm{Lagr}(\mathcal{C}, Q)$, so that X is given by a diagram

$$X_0 \leftarrow X_{01} \to 0$$

Let $L \in \mathrm{Lagr}(\mathcal{C}, Q)$ be the diagram

$$X_{01} \leftarrow X_{01} \to 0.$$

There is a canonical map $L \to X$. Note that $Q_{\mathrm{Lagr}}(L) = Q(0) \times_{Q(X_{01})} Q(X_{01}) \simeq 0$, so that the restriction of q to L is canonically nullhomotopic. A simple calculation shows that L is a Lagrangian in X, so that (X, q) represents $0 \in L_0(\mathrm{Lagr}(\mathcal{C}, Q), Q_{\mathrm{Lagr}})$. □

We now discuss the classification of *quadratic objects* of $(\mathcal{C}, Q)$ which are not necessarily Poincare. Let us begin with an example for motivation. Let $\mathcal{C} = \mathcal{D}^{\mathrm{perf}}(\mathbf{Z})$, let B be the symmetric bilinear functor associated to the duality functor

$$P_\bullet \mapsto \mathrm{Hom}(P_\bullet, \mathbf{Z}[-n])$$

and let $Q(X) = B(X, X)^{h\Sigma_2}$ be the associated quadratic functor. Let M be a compact oriented n-manifold with boundary. Then we have an intersection form

$$C^*(M, \partial M) \otimes C^*(M, \partial M) \stackrel{[M]}{\to} \mathbf{Z}[-n]$$

which determines a quadratic object $(C^*(M, \partial M), q_M)$ of $\mathcal{C}$. If M is a closed manifold, then this object is Poincare. In general it is not: the point q_M determines a map $C^*(M, \partial M) \to \mathbb{D}C^*(M, \partial M) \simeq C^*(M)$, which can be identified with the standard inclusion of relative cochains into cochains. The cofiber of this map is $C^*(\partial M)$. Note that this cofiber is itself described as a complex of cochains on a manifold (of dimension $n-1$ rather than n), and so determines a Poincare object of $(\mathcal{C}, \Sigma Q)$. We will now show that this is a general phenomenon.

So far, we have not used the full strength of our assumption that Q is a *quadratic* functor. The next result remedies this:

Proposition 5. *Suppose we are given a fiber sequence*

$$X' \to X \to X''$$

in $\mathcal{C}$. Then $Q(X'')$ is the total homotopy fiber of the diagram

$$\begin{array}{ccc} Q(X) & \longrightarrow & Q(X') \\ \downarrow & & \downarrow \\ B(X', X) & \longrightarrow & B(X', X') \end{array}$$

In other words, $Q(X'')$ is equivalent to the fiber of the map

$$Q(X') \times_{B(X',X')} B(X', X).$$

Remark 6. Let us state this result more informally. Think of $Q(X)$ as a spectrum parametrizing "quadratic forms" on X. The Proposition addresses the following question: given a quadratic form on X, when does it descend to $X'' = X/X'$? An obvious necessary condition is that it should vanish on X'. This implies that the associated bilinear form vanishes on X'. But we need a bit more: namely, to know that X' lies in the kernel of the associated bilinear form.

Proof. We wish to show that for every quadratic functor Q on $\mathcal{C}$, the triangle

$$Q(X'') \to Q(X) \to Q(X') \times_{B(X',X')} B(X, X')$$

is a fiber sequence. We have a fiber sequence of functors

$$Q_0(Z) \to Q(Z) \to B(Z, Z)^{h\Sigma_2}.$$

It therefore suffices to prove the result after replacing Q by Q_0 or $B(Z, Z)^{h\Sigma_2}$. In the first case, the polarization of Q_0 vanishes and the desired result follows from the fact that Q_0 is exact. Let us therefore assume that $Q(Z) = B(Z, Z)^{h\Sigma_2}$. Since B is exact in each variable, we can identify $B(X'', X'')$ with the total homotopy fiber of the diagram

$$\begin{array}{ccc} B(X, X) & \longrightarrow & B(X, X') \\ \downarrow & & \downarrow \\ B(X', X) & \longrightarrow & B(X', X'). \end{array}$$

In other words, $B(X'', X'')$ is the fiber of the map

$$B(X, X) \to B(X, X') \times_{B(X',X')} B(X', X) \simeq (B(X, X') \times B(X', X)) \times_{B(X',X') \times B(X',X')} B(X', X').$$

Taking homotopy fixed points with respect to Σ_2, we get a fiber sequence

$$Q(X'') \to Q(X) \to B(X, X') \times_{B(X',X')} Q(X').$$

□

Let us rewrite the fiber sequence of Proposition 5 as

$$Q(X') \times_{B(X',X')} B(X',X) \to \Sigma Q(X'') \to \Sigma Q(X).$$

Suppose that (Y,q) is a quadratic object of $(\mathcal{C},Q)$. There is a canonical point $\eta \in \Omega^\infty B(Y,\mathbb{D}Y)$ (corresponding to the identity map from Y to itself). Then q determines a point of $\Omega^\infty B(Y,Y)$, which is the image of η under the map $B(Y,\mathbb{D}Y) \to B(Y,Y)$ for some essentially unique map $u : Y \to \mathbb{D}Y$. Form a fiber sequence $Y \stackrel{u}{\to} \mathbb{D}(Y) \to \mathbb{D}(Y)/Y$. Then the pair (q,η) determines a point of $\Omega^\infty(Q(Y) \times_{B(Y,Y)} B(Y,\mathbb{D}Y)$. According to the above analysis, this is the 0th space of the fiber of the map of spectra

$$\Sigma Q(\mathbb{D}(Y)/Y) \to \Sigma Q(\mathbb{D}Y).$$

In particular, q determines a quadratic object of $(\mathcal{C},\Sigma Q)$, which we will denote by $(\mathbb{D}Y/Y,\overline{q})$.

The point $\overline{q}$ determines a map

$$v : \mathbb{D}Y/Y \to \Sigma\mathbb{D}(\mathbb{D}Y/Y) \simeq \Sigma\,\mathrm{fib}(\mathbb{D}^2(Y) \to \mathbb{D}Y) \simeq \Sigma\,\mathrm{fib}(u) \simeq \mathrm{cofib}(u) = \mathbb{D}Y/Y.$$

Unwinding the definitions, one sees that this is the identity map (up to a sign). Consequently, $(\mathbb{D}(Y)/Y,\overline{q})$ is a Poincare object of $(\mathcal{C},\Sigma Q)$. We have a canonical map $w : \mathbb{D}(Y) \to \mathbb{D}(Y)/Y$, and a canonical nullhomotopy of the image of $\overline{q}$ in $Q(\mathbb{D}(Y))$. This data determines a map

$$\mathbb{D}(Y) \to (\Sigma\mathbb{D})(\mathrm{cofib}(W)) \simeq \Sigma\mathbb{D}\Sigma Y \simeq \Sigma\Omega\mathbb{D}(Y) \simeq \mathbb{D}(Y),$$

which is also the identity map (up to sign). Consequently, we can regard $\mathbb{D}(Y)$ as a Lagrangian in $(\mathbb{D}(Y)/Y,\overline{q})$.

We can summarize our discussion as follows:

(∗) Given a quadratic object (Y,q) of $(\mathcal{C},Q)$, we can construct a Poincare object $(\mathbb{D}(Y)/Y,\overline{q})$ of $(\mathcal{C},\Sigma Q)$ and a Lagrangian $\mathbb{D}(Y)$ in $(\mathbb{D}(Y)/Y,\overline{q})$.

Remark 7. The converse to (∗) is true as well: given a Poincare object $(Z,\overline{q})$ in $(\mathcal{C},Q)$ and a Lagrangian $f : L \to Z$, we can equip the fiber $\mathrm{fib}(f)$ with the structure of a quadratic object of $(\mathcal{C},Q)$.

Let us now suppose that $\mathcal{C}_0 \subseteq \mathcal{C}$ is a stable subcategory which is closed under duality. Suppose we are given a quadratic object (Y,q) of $(\mathcal{C},Q)$ whose image in $\mathcal{C}/\mathcal{C}_0$ is a Poincare object. Then the canonical map $u : Y \to \mathbb{D}(Y)$ becomes invertible in $\mathcal{C}/\mathcal{C}_0$, so the cofiber $\mathrm{cofib}(u) = \mathbb{D}(Y)/Y$ belongs to $\mathcal{C}_0$. Consequently, the above construction produces a Poincare object $(\mathbb{D}(Y)/Y,\overline{q})$ of $(\mathcal{C}_0,\Sigma Q')$ (where $Q' = Q|\,\mathcal{C}_0$). Suppose that this Poincare object is nullcobordant: that is, we can choose a Lagrangian $L \to \mathbb{D}(Y)/Y$ in $(\mathcal{C}_0,\Sigma Q')$. Then $(\mathbb{D}(Y)/Y,\overline{q})$ has *two* Lagrangians in the ∞-category $\mathcal{C}$: L and $\mathbb{D}(Y)$. Each of these provides a cobordism of $(\mathbb{D}(Y)/Y,\overline{q})$ with the zero object. We have seen that cobordisms can be composed: if we compose these cobordisms, we obtain a cobordism $L \times_{\mathbb{D}(Y)/Y} \mathbb{D}(Y)$ from the zero Poincare object of $(\mathcal{C},\Sigma Q)$ to itself. Such a cobordism can be regarded as a Poincare object (Y',q') of $(\mathcal{C},\Omega\Sigma Q) = (\mathcal{C},Q)$. Moreover, since L vanishes in $\mathcal{C}/\mathcal{C}_0$, we note that (Y',q') and (Y,q) determine the same quadratic object of $\mathcal{C}/\mathcal{C}_0$.

Remark 8. Let (Y_0,q_0) be any Poincare object of $\mathcal{C}/\mathcal{C}_0$. By construction, we can always lift (Y_0,q_0) to a quadratic object (Y,q) of $\mathcal{C}$. The above discussion shows that we can adjust (Y,q) to be a Poincare object of $\mathcal{C}$ if and only if a certain obstruction in $L_0(\mathcal{C}_0,\Sigma Q')$ vanishes. One we have proven the theorem, we can identify this obstruction with the image of (Y_0,q_0) under the boundary map

$$\pi_1 L(\mathcal{C}/\mathcal{C}_0,\Sigma Q'') \to \pi_0 L(\mathcal{C}_0,\Sigma Q')$$

determined by the fiber sequence of spaces

$$L(\mathcal{C}_0,\Sigma Q') \to L(\mathcal{C},\Sigma Q) \to L(\mathcal{C}/\mathcal{C}_0,\Sigma Q'').$$

We now turn to the proof of Theorem 1. We must show that every point η of

$$\mathrm{Poinc}(\mathcal{C} \,/\, \mathcal{C}_0, Q'')_\bullet(\Delta^n) \times_{\mathrm{Poinc}(\mathcal{C} \,/\, \mathcal{C}_0, Q'')_\bullet(\Lambda^n_i)} \mathrm{Poinc}(\mathcal{C}, Q)_\bullet(\Lambda^n_i)$$

can be lifted (up to homotopy) to a point of $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet(\Delta^n)$. The point η determines a Poincare object (X_0, q_0) of $(\mathcal{C} \,/\, \mathcal{C}_0)_{[n]}$. Here we think of X_0 as a contravariant functor from the collection of nonempty subsets of $\{0, \ldots, n\}$ into $\mathcal{C} \,/\, \mathcal{C}_0$. Let $\sigma = [n] = \{0, \ldots, n\}$ and let $\tau = [n] - \{i\} \subseteq \sigma$. Then η determines objects $X(S) \in \mathcal{C}$ lifting $X_0(S)$ for $S \notin \{\sigma, \tau\}$, together with a point of

$$q_1 \in \Omega^\infty \varprojlim_{S \notin \{\sigma, \tau\}} Q(X(S))$$

which is nondegenerate and compatible with q_0. Using the construction of $\mathcal{C} \,/\, \mathcal{C}_0$ and Q'', we can extend X to a functor defined on all nonempty subsets of $[n]$ and q_1 to a point $q \in \Omega^\infty Q_{[n]}(X)$ (compatible with q_0). What is not clear is that (X, q) is a Poincare object of $(\mathcal{C}_{[n]}, Q_{[n]})$. The point q determines a map $X \to \mathbb{D}_{[n]}(X)$, which may fail to be invertible when evaluated at σ and τ.

Let $\mathcal{D}$ be the full subcategory of $\mathcal{C}_{[n]}$ spanned by those functors Z such that $Z(S) \simeq 0$ for $S \notin \{\sigma, \tau\}$, and $Z(\sigma), Z(\tau) \in \mathcal{C}_0$. We can identify objects of $\mathcal{D}$ with morphisms $Z(\sigma) \to Z(\tau)$ in $\mathcal{C}_0$, and so have an equivalence of ∞-categories $\mathcal{D} \simeq \mathrm{Lagr}(\mathcal{C}_0, Q')$. When $n = 1$, the quadratic functor $Q_{[n]}| \, \mathcal{D}^{op}$ is precisely the quadratic functor Q_{Lagr} appearing in Lemma 3. In the general case, a simple calculation gives $Q_{[n]}| \, \mathcal{D}^{op} \simeq \Omega^{n-1} Q_{\mathrm{Lagr}}$.

Let Y denote the cofiber of the map $u : X \to \mathbb{D}_{[n]}(X)$, so that Y has the structure of a Poincare object of $(\mathcal{D}, (\Sigma Q_{[n]})| \, \mathcal{D}^{op}) = (\mathrm{Lagr}(\mathcal{C}, Q), \Omega^{n-2} Q_{\mathrm{Lagr}})$. Invoking Lemma 3, we deduce that every Poincare object of $(\mathcal{D}, (\Sigma Q_{[n]})| \, \mathcal{D})$ is nullcobordant. In particular, Y is nullcobordant. Choosing a Lagrangian in Y, we obtain a procedure for modifying (X, q) to obtain a Poincare object of $(\mathcal{C}_{[n]}, Q_{[n]})$, which gives the desired lift of η.