Localization (Lecture 8)

February 10, 2011

Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$. In the last lecture, we asserted without proof that the simplicial space $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$ satisfies the Kan condition. Our goal in this lecture is to formulate a generalization of this assertion, which we will prove in the next lecture.

We begin with some generalities. Let $\mathcal{J}$ be an ∞-category. We say that $\mathcal{J}$ is *filtered* if it satisfies the following conditions:

- $\mathcal{J}$ is nonempty.
- For every pair of objects $X, Y \in \mathcal{J}$, there is a third object $Z \in \mathcal{J}$ and a pair of maps $X \to Z \leftarrow Y$.
- For every pair of objects $X, Y \in \mathcal{J}$ and every map of spaces $S^n \to \mathrm{Map}_{\mathcal{J}}(X, Y)$, there is a map $g : Y \to Z$ such that the composite map $S^n \to \mathrm{Map}_{\mathcal{J}}(X, Y) \to \mathrm{Map}_{\mathcal{J}}(X, Z)$ is nullhomotopic.

Let $\mathcal{C}$ be an ∞-category. By a *filtered diagram* in $\mathcal{C}$ we will refer to a functor $\mathcal{J}^{op} \to \mathcal{C}$, where $\mathcal{J}$ is a filtered ∞-category. We will denote a filtered diagram in $\mathcal{C}$ by (X_α), where each X_α is an object of $\mathcal{C}$. The collection of filtered diagrams in $\mathcal{C}$ can be organized into an ∞-category $\mathrm{Pro}(\mathcal{C})$, where morphism spaces are given by

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}((X_\alpha), (Y_\beta)) = \varprojlim_{\beta} \varinjlim_{\alpha} \mathrm{Map}_{\mathcal{C}}(X_\alpha, Y_\beta).$$

We refer to the objects of $\mathrm{Pro}(\mathcal{C})$ as *Pro-objects* of $\mathcal{C}$.

We will identify $\mathcal{C}$ with a full subcategory of $\mathrm{Pro}(\mathcal{C})$ (each object $X \in \mathcal{C}$ determines a filtered diagram (X) indexed by the one-point ∞-category $*$). For every filtered diagram (X_α), we can identify the corresponding object of $\mathrm{Pro}(\mathcal{C})$ with the (homotopy) limit $\varprojlim X_\alpha$ in $\mathrm{Pro}(\mathcal{C})$. We can think of $\mathrm{Pro}(\mathcal{C})$ as the ∞-category obtained from $\mathcal{C}$ by formally adjoining limits of filtered diagrams. In fact, $\mathrm{Pro}(\mathcal{C})$ has the following universal property: if $\mathcal{D}$ is an ∞-category which admits filtered limits, then the ∞-category of functors $\mathrm{Pro}(\mathcal{C}) \to \mathcal{D}$ which preserve filtered limits is equivalent to the ∞-category of functors $\mathcal{C} \to \mathcal{D}$.

Remark 1. It is not necessary to allow arbitrary filtered ∞-categories in the definition of $\mathrm{Pro}(\mathcal{C})$. One can show that every filtered diagram is equivalent (in the ∞-category $\mathrm{Pro}(\mathcal{C})$) to a diagram indexed by a filtered partially ordered set.

Remark 2. If $\mathcal{C}$ is a stable ∞-category, then the ∞-category $\mathrm{Pro}(\mathcal{C})$ is also stable.

Suppose that $\mathcal{C}$ is a stable ∞-category and let $\mathcal{C}_0$ be a stable subcategory of $\mathcal{C}$ (that is, a subcategory closed under the formation of fibers and cofibers). We will say that a map $f : X' \to X$ in $\mathcal{C}$ is a $\mathcal{C}_0$-*equivalence* if the cofiber $\mathrm{cofib}(f)$ belongs to $\mathcal{C}_0$. If we regard X as fixed, then the collection of all $\mathcal{C}_0$-equivalences $f_\alpha : X_\alpha \to X$ forms a filtered ∞-category (in fact, it is an ∞-category which admits finite limits). Consequently, we can regard (X_α) as a Pro-object of $\mathcal{C}$. We will denote this pro-object by $I(X) \in \mathrm{Pro}(\mathcal{C})$.

Definition 3. Let $\mathcal{C}$ be a (small) stable ∞-category and let $\mathcal{C}_0$ be a stable subcategory of $\mathcal{C}$. We let $\mathcal{C} / \mathcal{C}_0$ denote the full subcategory of $\mathrm{Pro}(\mathcal{C})$ spanned by objects of the form $I(X)$, where $X \in \mathcal{C}$.

It is not difficult to see that the construction $X \mapsto I(X)$ commutes with finite limits. From this, one can deduce that $\mathcal{C} / \mathcal{C}_0$ is closed under passing to fibers in $\mathrm{Pro}(\mathcal{C})$. It follows that $\mathcal{C} / \mathcal{C}_0$ is a stable subcategory of $\mathrm{Pro}(\mathcal{C})$.

The following result justifies our notation:

Proposition 4. *Let $\mathcal{D}$ be a stable ∞-category. Then composition with the functor $I : \mathcal{C} \to \mathcal{C} / \mathcal{C}_0$ induces an equivalence from the ∞-category of exact functors $f : \mathcal{C} / \mathcal{C}_0 \to \mathcal{D}$, to the ∞-category of exact functors $F : \mathcal{C} \to \mathcal{D}$ such that $F | \mathcal{C}_0$ is trivial.*

Proof. Note that if $X \in \mathcal{C}_0$, then the filtered ∞-category of $\mathcal{C}_0$-equivalences $X' \to X$ has an initial object (namely, the map $0 \to X$), so that $I(X) \simeq 0$. It follows that for any exact functor $f : \mathcal{C} / \mathcal{C}_0 \to \mathcal{D}$, the composition $F = f \circ I$ is an exact functor from $\mathcal{C}$ to $\mathcal{D}$ which annihilates $\mathcal{C}_0$.

We now produce an inverse to the preceding construction. Embed $\mathcal{D}$ in a stable ∞-category $\overline{\mathcal{D}}$ which admits filtered limits (for example, we can take $\overline{\mathcal{D}} = \mathrm{Pro}(\mathcal{D})$). Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor which annhilates $\mathcal{C}_0$. Then F extends to a functor $\overline{F} : \mathrm{Pro}(\mathcal{C}) \to \overline{\mathcal{D}}$ which commutes with filtered limits. For each $X \in \mathcal{C}$, we have a canonical map $I(X) \to X$ in $\mathrm{Pro}(\mathcal{C})$, hence a map $u : \overline{F}(I(X)) \to F(X)$ in $\overline{\mathcal{D}}$. The cofiber of the map $I(X) \to X$ is a filtered limit of objects of $\mathcal{C}_0$. Since F annihilates $\mathcal{C}_0$ and $\overline{F}$ commutes with filtered limits, we deduce that u is invertible: that is, we can write $F = \overline{F} \circ I$. In particular, $\overline{F}$ carries $I(\mathcal{C}) = \mathcal{C} / \mathcal{C}_0$ into the subcategory $\mathcal{D} \subseteq \overline{\mathcal{D}}$. Let us denote this restricted functor by $f : \mathcal{C} / \mathcal{C}_0 \to \mathcal{D}$; then $F = f \circ I$ as desired. □

Now suppose that $\mathcal{C}$ is equipped with a nondegenerate quadratic functor Q. We can extend Q to a functor $\widehat{Q} : \mathrm{Pro}(\mathcal{C})^{op} \to \mathrm{Sp}$ by the formula

$$\widehat{Q}((X_\alpha)) = \varinjlim_{\alpha} Q(X_\alpha).$$

It is easy to see that $\widehat{Q}$ is a quadratic functor on $\mathrm{Pro}(\mathcal{C})$, whose polarization $\widehat{B}$ is given by the formula

$$\widehat{B}((X_\alpha), (Y_\beta)) = \varinjlim_{\alpha, \beta} B(X_\alpha, Y_\beta)$$

where B denotes the polarization of Q.

Definition 5. Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$. Let B be the polarization of Q and let $\mathbb{D}$ denote the corresponding duality functor. We will say that Q is *compatible* with a stable subcategory $\mathcal{C}_0$ if the duality functor $\mathbb{D}$ carries $\mathcal{C}_0$ to itself. In this case, $Q' = Q | \mathcal{C}_0^{op}$ is a nondegenerate quadratic functor on $\mathcal{C}_0$, having polarization $B' = B | \mathcal{C}_0^{op} \times \mathcal{C}_0^{op}$ and duality functor $\mathbb{D}' = \mathbb{D} | \mathcal{C}_0$.

In the above situation, the composition

$$\mathcal{C} \xrightarrow{\mathbb{D}} \mathcal{C}^{op} \to (\mathcal{C} / \mathcal{C}_0)^{op}$$

annihilates the subcategory $\mathcal{C}_0$, and therefore factors (in an essentially unique way) as a composition

$$\mathcal{C} \to \mathcal{C} / \mathcal{C}_0 \xrightarrow{\mathbb{D}''} (\mathcal{C} / \mathcal{C}_0)^{op}.$$

Using the fact that $\mathbb{D}$ has order 2, we deduce easily that $\mathbb{D}''$ also has order 2; in particular, it is a contravariant equivalence of $\mathcal{C} / \mathcal{C}_0$ with itself.

Proposition 6. *In the above situation, the quadratic functor $\widehat{Q}$ on $\mathrm{Pro}(\mathcal{C})$ restricts to a nondegenerate quadratic functor Q'' on $\mathcal{C} / \mathcal{C}_0$, whose duality functor is given by $\mathbb{D}''$.*

Proof. Let X be an object of $\mathcal{C}$ and write $I(X) = (X_\alpha)$. Note that if $Z \in \mathcal{C}_0$, then any map $I(X) \to Z$ in $\mathrm{Pro}(\mathcal{C})$ is nullhomotopic: such a map must factor through some X_α, but the fiber F of the induced map $X_\alpha \to Z$ also belongs to the filtered system (X_α) (and composition $F \to X_\alpha \to Z$ is nullhomotopic).

Let $Y \in \mathcal{C}$ and write $I(Y) = (Y_\beta)$. The above argument shows that

$$\mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), Y_\beta) \to \mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), Y)$$

for each index β, so that $\mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), I(Y)) \simeq \mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), Y)$. We therefore obtain a canonical homotopy equivalence

$$\begin{aligned}\mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), \mathbb{D}'' I(Y)) &\simeq \mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), I(\mathbb{D}Y)) \\ &\simeq \mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), \mathbb{D}Y) \\ &\simeq \varinjlim_{\alpha} \mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(X_\alpha, \mathbb{D}Y) \\ &\simeq \varinjlim_{\alpha} B(X_\alpha, Y).\end{aligned}$$

For every index β, the cofiber Z of the map $Y_\beta \to Y$ belongs to $\mathcal{C}_0$, so that $\mathbb{D}(Z) \in \mathcal{C}_0$. It follows that

$$\varinjlim_{\alpha} B(X_\alpha, Z) \simeq \mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), \mathbb{D}(Z)) \simeq 0,$$

so that

$$\varinjlim_{\alpha} B(X_\alpha, Y) \simeq \varinjlim_{\alpha} B(X_\alpha, Y_\beta).$$

Passing to the limit over β, we get

$$\mathrm{Mor}_{\mathrm{Pro}(\mathcal{C})}(I(X), \mathbb{D}'' I(Y)) \simeq \varinjlim_{\alpha} B(X_\alpha, Y) \simeq \varinjlim_{\alpha,\beta} B(X_\alpha, Y_\beta) = \widehat{B}(I(X), I(Y)),$$

so that $\mathbb{D}''$ is the duality functor associated to the bilinear pairing $\widehat{B}$ restricted to $\mathcal{C} / \mathcal{C}_0$. $\square$

Let (X, q) be a quadratic object of $\mathcal{C}$. Then q determines a point $q'' \in \Omega^\infty \widehat{Q}(I(X))$, so that $(I(X), q'')$ can be viewed as a quadratic object of $(\mathcal{C} / \mathcal{C}_0, Q'')$. We have just verified that the functor $I : \mathcal{C} \to \mathcal{C} / \mathcal{C}_0$ interchanges the duality functors induced by Q and Q'' respectively. It follows that if (X, q) is a Poincare object of $(\mathcal{C}, Q)$, then $(I(X), q'')$ is a Poincare object of $(\mathcal{C} / \mathcal{C}_0, Q'')$. This construction determines a map of classifying spaces

$$\mathrm{Poinc}(\mathcal{C}, Q) \to \mathrm{Poinc}(\mathcal{C} / \mathcal{C}_0, Q'').$$

Suppose that $\mathcal{C}_0$ is closed under the formation of direct summands in $\mathcal{C}$. Then the fiber of this map (over the zero object) can be identified with $\mathrm{Poinc}(\mathcal{C}_0, Q')$. Applying the same reasoning to the ∞-categories $\mathcal{C}_{[n]}$ for $n \geq 0$, we obtain a fiber sequence of simplicial spaces

$$\mathrm{Poinc}(\mathcal{C}_0, Q')_\bullet \to \mathrm{Poinc}(\mathcal{C}, Q)_\bullet \xrightarrow{\phi} \mathrm{Poinc}(\mathcal{C} / \mathcal{C}_0, Q'')_\bullet$$

We will prove the following result in the next lecture:

Theorem 7. *Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor Q, and let $\mathcal{C}_0$ be a stable subcategory of $\mathcal{C}$ which is closed under duality. Then the map $\phi : \mathrm{Poinc}(\mathcal{C}, Q)_\bullet \to \mathrm{Poinc}(\mathcal{C} / \mathcal{C}_0, Q'')_\bullet$ is a Kan fibration of simplicial spaces.*

Corollary 8. *Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor Q. Then the simplicial space $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$ satisfies the Kan condition.*

Proof. Apply Theorem 7 in the special case $\mathcal{C}_0 = \mathcal{C}$. $\square$

Corollary 9. *In the situation of Theorem 7, suppose that every direct summand of an object of* $\mathcal{C}_0$ *also lies in* $\mathcal{C}_0$*. Then we have a fiber sequence of L-theory spaces*

$$L(\mathcal{C}_0, Q') \to L(\mathcal{C}, Q) \to L(\mathcal{C} / \mathcal{C}_0, Q''),$$

and therefore a long exact sequence of abelian groups

$$\cdots \to L_1(\mathcal{C} / \mathcal{C}_0, Q'') \to L_0(\mathcal{C}_0, Q') \to L_0(\mathcal{C}, Q) \to L_0(\mathcal{C} / \mathcal{C}_0, Q'').$$

Proof. Combine Theorem 7 with the result stated at the end of the previous lecture. □