Simplicial Spaces (Lecture 7)

February 6, 2011

Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$. In the last lecture, we defined an L-theory space $L(\mathcal{C}, Q)$, whose path components comprise the abelian group $L_0(\mathcal{C}, Q)$ of Lecture 5. We would like to understand the homotopy type $L(\mathcal{C}, Q)$ better. For example, we might ask for an interpretation of the higher homotopy groups $L_n(\mathcal{C}, Q) = \pi_n L(\mathcal{C}, Q)$.

By definition, $L(\mathcal{C}, Q)$ is given to us as the geometric realization of a simplicial space $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$. In general, it is not easy to describe the homotopy groups of a geometric realization even if the homotopy groups of the indiviual terms are well-understood (for example, it is hard to describe the homotopy groups of the geometric realization of the simplicial set $\partial\, \Delta^3$).

For a general simplicial space $X_\bullet$, there are two face maps $d_0, d_1 : X_1 \to X_0$ which induce a map $\pi_0(X_1) \to \pi_0 X_0 \times \pi_0 X_0$. The image of this map is a relation R on $\pi_0 X_0$, and the quotient of $\pi_0 X_0$ by the equivalence relation generated by R can be identified with $\pi_0 |X_\bullet|$. However, in our case R is the relation of cobordism of Poincare objects, which is already an equivalence relation. This is a special feature of $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$ which makes the homotopy group $\pi_0 L(\mathcal{C}, Q)$ easier to compute. We would like to generalize this observation.

We begin by introducing some notation.

Definition 1. Let $\mathbf{\Delta}$ denote the category of *combinatorial simplices*: that is, nonempty finite linearly ordered sets of the form $\{0, \ldots, n\}$. In this lecture, we will identify the objects of $\mathbf{\Delta}$ with the corresponding simplicial sets $\Delta^0, \Delta^1, \cdots$. A *simplicial space* is a functor from $\mathbf{\Delta}^{op}$ to the ∞-category of spaces. If X is a simplicial space, we will denote the individual spaces of X by $X(\Delta^0)$, $X(\Delta^1)$, and so forth.

If X is a simplicial space, then X determines a functor from the ordinary category of simplicial sets into the ∞-category of spaces, given by

$$K \mapsto \varprojlim_{\sigma : \Delta^n \to K} X(\Delta^n).$$

We will denote this functor by $K \mapsto X(K)$. (More abstractly: we regard X as a functor defined on all simplicial sets, rather than just standard simplices, by taking a right Kan extension.)

Remark 2. We can identify $X(K)$ with the space of maps from K to X in the ∞-category of simplicial spaces (where we regard K as a simplicial space by endowing it with the discrete topology in each degree).

Definition 3. Let $f : X \to Y$ be a map of simplicial spaces. We will say that f is a *Kan fibration* if the following condition is satisfied: for $0 \leq i \leq n$, the map

$$X(\Delta^n) \to X(\Lambda^n_i) \times_{Y(\Lambda^n_i)} Y(\Delta^n)$$

is surjective on connected components (here the fiber product denotes a homotopy fiber product). We will say that f is a *trivial Kan fibration* if, for each $n \geq 0$, the map

$$X(\Delta^n) \to X(\partial\, \Delta^n) \times_{Y(\partial\, \Delta^n)} Y(\Delta^n).$$

We will say that a simplicial space X *satisfies the Kan condition* if the map $X \to *$ is a Kan fibration, where $*$ denotes the constant simplicial space with value equal to a single point.

Remark 4. When we restrict our attention to simplicial sets (which we regard as a special case of simplicial spaces), Definition 3 recovers the usual notion of Kan fibration, trivial Kan fibration, and Kan complex.

If X is a simplicial space satisfying the Kan condition, then the surjectivity of the map $\pi_0 X(\Delta^2) \to \pi_0 X(\Lambda^2_1)$ guarantees that the image of $\pi_0 X(\Delta^1)$ is an equivalence relation on $\pi_0 X(\Delta^0)$. Since we know that the latter condition holds for $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$, we are naturally led to conjecture the following:

Theorem 5. *The simplicial space* $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$ *satisfies the Kan condition.*

We will prove Theorem 5 later this week. The remainder of this lecture is devoted to exploring some consequences of Theorem 5.

Recall that if $f : X \to Y$ is a trivial Kan fibration of simplicial sets, then f induces a homotopy equivalence of geometric realizations $|X| \to |Y|$. This generalizes to simplicial spaces:

Proposition 6. *Let $f : X \to Y$ be a trivial Kan fibration of simplicial spaces. Then the induced map $|X| \to |Y|$ is a homotopy equivalence.*

Proof. The category $\mathcal{S}\mathrm{et}_\Delta$ of simplicial sets is a model for the ∞-category of spaces. We may therefore choose a simplicial object $\overline{X}$ of the category of simplicial sets representing X, and a simplicial object $\overline{Y}$ of the category of simplicial sets representing Y, such that f is modelled by a map of bisimplicial sets $\overline{f} : \overline{X} \to \overline{Y}$. Without loss of generality, we may assume that $\overline{X}$ and $\overline{Y}$ are Reedy fibrant and that $\overline{f}$ is a Reedy fibration. Then each of the maps

$$X(\Delta^n) \to X(\partial \Delta^n) \times_{Y(\partial \Delta^n)} Y(\Delta^n)$$

is modelled by a Kan fibration of simplicial sets

$$\overline{X}(\Delta^n) \to \overline{X}(\partial \Delta^n) \times_{\overline{Y}(\partial \Delta^n)} \overline{Y}(\Delta^n)$$

Our assumption on f guarantees that this map is surjective on connected components. Since it is a Kan fibration, it is surjective on simplices of every dimension. In other words, we deduce that for each $m \geq 0$, the map of simplicial sets

$$\overline{X}_m \to \overline{Y}_m$$

is a trivial Kan fibration. In particular, the map of bisimplicial sets $\overline{f} : \overline{X} \to \overline{Y}$ is a levelwise homotopy equivalence in the "horizontal" direction, and so induces a homotopy equivalence after geometric realization. □

Proposition 7. *Let Y be a simplicial space. Then there exists a simplicial set X and a trivial Kan fibration $f : X \to Y$.*

Proof. We successively build n-skeletal simplicial sets $\mathrm{sk}^n X$ and maps $\mathrm{sk}^n X \to Y$ such that the maps

$$\mathrm{sk}^n X(\Delta^m) \to (\mathrm{sk}^n X)(\partial \Delta^m) \times_{Y(\partial \Delta^m)} Y(\Delta^m)$$

are surjective on connective components for $m \leq n$. Assume that $\mathrm{sk}^{n-1} X$ has already been constructed. Let S be the set of connected components of the fiber product

$$(\mathrm{sk}^{n-1} X)(\partial \Delta^n) \times_{Y(\partial \Delta^n)} Y(\Delta^n).$$

and let $\mathrm{sk}^n X$ be the simplicial set obtained from $\mathrm{sk}^{n-1} X$ by adjoining one nondegenerate n-simplex for every element of S (with the obvious attaching maps). There is an evident map of simplicial spaces $\mathrm{sk}^n X \to Y$ having the desired properties. □

Let Y be a simplicial space satisfying the Kan condition, and suppose that we wish to describe the homotopy groups of the geometric realization $|Y|$. Choose a trivial Kan fibration $X \to Y$, where X is a simplicial set. Then the map $|X| \to |Y|$ is a homotopy equivalence, so the homotopy groups of $|Y|$ are the same as the homotopy groups of $|X|$. Moreover, since $X \to Y$ is a Kan fibration, the simplicial set X also satisfies the Kan condition: that is, it is a Kan complex in the usual sense. Let us fix a base point x of $X(\Delta^0)$ (which determines a base point in $Y(\Delta^0)$) and compute all homotopy groups with respect to that base point. If K is a simplicial set with a simplicial subset K_0, let $Y(K, K_0)$ denote the homotopy fiber of the map $Y(K) \to Y(K_0)$ (over the point determined by the base point), and define $X(K, K_0)$ similarly.

Because X is a Kan complex, there is a simple combinatorial recipe for extracting the homotopy groups $\pi_n|X|$. Let us recall how this goes. Every class in $\pi_n|X|$ is represented by a point $\eta \in X(\Delta^n, \partial\,\Delta^n)$. Let $K \subseteq \partial\,\Delta^{n+1}$ be the subset obtained by removing the interiors of two faces, so that we have a canonical bijection

$$X(\partial\,\Delta^{n+1}, K) \to X(\Delta^n, \partial\,\Delta^n) \times X(\Delta^n, \partial\,\Delta^n).$$

A pair of elements $\eta, \eta' \in X(\Delta^n, \partial\,\Delta^n)$ determine the same element in $\pi_n|X|$ if and only if the corresponding element of $X(\partial\,\Delta^{n+1}, K)$ can be lifted to $X(\Delta^{n+1}, K)$.

Since $X \to Y$ is a trivial Kan fibration, the map

$$\phi : X(\Delta^n, \partial\,\Delta^n) \to Y(\Delta^n, \partial\,\Delta^n).$$

is surjective on connected components: that is, every element of $\pi_0 Y(\Delta^n, \partial\,\Delta^n)$ comes from a point $\eta \in X(\Delta^n, \partial\,\Delta^n)$. Suppose we are given a pair of points of $Y(\Delta^n, \partial\,\Delta^n)$, given by the images of elements $\eta, \eta' \in X(\Delta^n, \partial\,\Delta^n)$. This pair of points determines a point $\zeta \in X(\partial\,\Delta^{n+1}, K)$ having image $\zeta_0 \in Y(\partial\,\Delta^{n+1}, K)$. Since the map

$$X(\Delta^{n+1}) \to X(\partial\,\Delta^{n+1}) \times_{Y(\partial\,\Delta^{n+1})} Y(\Delta^{n+1})$$

is surjective on connected components, we deduce that ζ_0 lifts to a point of $Y(\Delta^{n+1})$ if and only if ζ lifts to a point of $X(\Delta^{n+1})$. We have proven the following:

Proposition 8. *Let Y be a simplicial space satisfying the Kan condition, and choose a base point $y \in Y(\Delta^0)$ (so that we can regard Y as a simplicial pointed space). Then $\pi_n|Y|$ can be identified with the quotient of the set $\pi_0 Y(\Delta^n, \partial\,\Delta^n)$ by the following equivalence relation: two homotopy classes $[\eta], [\eta'] \in \pi_0 Y(\Delta^n, \partial\,\Delta^n)$ represent the same class in $\pi_n|Y|$ if and only if the corresponding point of $Y(\partial\,\Delta^{n+1}, K)$ lifts to a point of $Y(\Delta^{n+1})$.*

Let us now apply this analysis to the case of interest, where Y is the simplicial space $\mathrm{Poinc}(\mathcal{C}, Q)_\bullet$. Unwinding the definitions, we see that $Y(\Delta^n, \partial\,\Delta^n)$ is a classifying space for Poincare objects (X, q) of $\mathcal{C}_{[n]}$ (using the notation of the previous lecture) such that $X(S) \simeq 0$ for all proper subsets $S \subsetneq [n]$. In this case, X is determined by a single object $C = X([n]) \in \mathcal{C}$. Moreover, we have

$$Q_{[n]}(X) = \varprojlim_S Q(X(S)) = \varprojlim_S \begin{cases} Q(C) & \text{if } S = [n] \\ 0 & \text{otherwise.} \end{cases}$$

The relevant diagram is parametrizes by partially ordered set of faces of an n-simplex, taking the value 0 on every proper face. Consequently, the limit in question is given by $\Omega^n Q(C)$. We can summarize our analysis as follows:

- $(*)$ Let $Y = \mathrm{Poinc}(\mathcal{C}, Q)_\bullet$. Then $Y(\Delta^n, \partial\,\Delta^n)$ is a classifying space for Poincare objects of $(\mathcal{C}, \Omega^n Q)$.

Now suppose we are given two Poincare objects for $(\mathcal{C}, \Omega^n Q)$. They determine a point of $Y(\partial\,\Delta^{n+1})$: that is, a functor from the partially ordered set of all nonempty *proper* subsets of $[n+1]$ into $\mathcal{C}$. Moreover, this functor vanishes identically except on two subsets of $[n+1]$ of cardinality n. Unwinding the definitions, we see that lifting this data to a Poincare object of $\mathcal{C}_{[n+1]}$ is equivalent to specifying a *cobordism* betwee the corresponding Poincare object of $(\mathcal{C}, \Omega^n Q)$. We have proven the following:

Theorem 9. *The abelian group* $L_n(\mathcal{C}, Q) = \pi_n L(\mathcal{C}, Q)$ *is canonically isomorphic to* $L_0(\mathcal{C}, \Omega^n Q)$.

We close with a result that will be needed in the next lecture:

Proposition 10. *Let* $X \to Y \stackrel{u}{\to} Z$ *be a fiber sequence of simplicial spaces. Suppose that* u *is a Kan fibration. Then*

$$|X| \to |Y| \to |Z|$$

is a fiber sequence of spaces.

Proof. Choose a trivial Kan fibration $f : Z' \to Z$, where Z' is a simplicial set (and choose a base point of Z' lying over the chosen base point of Z). Now choose a trivial Kan fibration $g : Y' \to Y \times_Z Z'$, where Y' is a simplicial set. The canonical map $Y' \to Y$ is a composition of g with a pullback of f, and therefore a trivial Kan fibration. Let X' be the fiber of the map of simplicial sets $Y' \to Z'$. The canonical map $X' \to X$ is a pullback of g and therefore a trivial Kan fibration. We have a commutative diagram of fiber sequences

$$\begin{array}{ccccc} X' & \longrightarrow & Y' & \longrightarrow & Z' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

where the vertical maps are trivial Kan fibrations, and therefore induce homotopy equivalences after geometric realization. It will therefore suffice to prove that the sequence of spaces

$$|X'| \to |Y'| \to |Z'|$$

is a fiber sequence. Since these are simplicial sets, it suffices to prove that the map $Y' \to Z'$ is a Kan fibration. This map is given by the composition of g (a trivial Kan fibraation) with the projection map $u' : Y \times_Z Z' \to Z'$. It will therefore suffice to show that u' is a Kan fibration (of simplicial spaces). This is clear, since u' is a pullback of the Kan fibration u. □