L-theory Spaces (Lecture 6)

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Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$, which we regard as fixed throughout this lecture. We let $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$ denote the polarization of Q, and $\mathbb{D} : \mathcal{C}^{op} \to \mathcal{C}$ the corresponding duality functor. In the last lecture, we introduce the notion of a *cobordism* between two Poincare objects (X, q) and (X', q') of $\mathcal{C}$. We saw that cobordism is an equivalence relation and defined $L_0(\mathcal{C}, Q)$ to be the set of equivalence classes.

In this lecture, we would like to refine the invariant $L_0(\mathcal{C}, Q)$. We will accomplish this by defining an *L-theory space* $L(\mathcal{C}, Q)$, with $\pi_0 L(\mathcal{C}, Q) = L_0(\mathcal{C}, Q)$.

We first describe an approximation to this L-theory space. We let $\mathrm{Poinc}(\mathcal{C}, Q)$ denote a *classifying space for Poincare objects of* $\mathcal{C}$. That is, $\mathrm{Poinc}(\mathcal{C}, Q)$ is an ∞-category whose objects are Poincare objects (X, q) of $\mathcal{C}$, where a morphism from (X, q) to (X', q') is an isomorphism $\alpha : X \to X'$ together with a path joining q to the image of q' in the space $\Omega^\infty Q(X)$. $\mathrm{Poinc}(\mathcal{C}, Q)$ is an ∞-category in which every morphism is invertible and therefore a Kan complex. We will simply refer to $\mathrm{Poinc}(\mathcal{C}, Q)$ as a *space*. It is not the space we are looking for, because cobordant Poincare objects need not lie in the same connected component of $\mathrm{Poinc}(\mathcal{C}, Q)$.

Notation 1. Fix an integer $n \geq 0$. We let $\mathcal{F}_n$ denote the collection of nonempty subsets of the set $\{0, 1, \ldots, n\}$. We regard $\mathcal{F}_n$ as a partially ordered set with respect to inclusions. (It may be helpful to think of $\mathcal{F}_n$ as the partially ordered set of faces of the standard n-simplex Δ^n.) Note that $\mathcal{F}_n$ has a largest element, given by the set $[n] = \{0, \ldots, n\}$.

Let $\mathcal{C}_{[n]}$ denote the ∞-category of functors $\mathrm{Fun}(\mathcal{F}_n^{op}, \mathcal{C})$ from $\mathcal{F}_n^{op}$ into $\mathcal{C}$. We define a functor

$$Q_{[n]} : \mathcal{C}_{[n]}^{op} \to \mathrm{Sp}$$

by the formula $Q_{[n]}(X) = \varprojlim_{S \in \mathcal{F}_n} Q(X(S))$.

Using the fact that Q is quadratic, it follows immediately that $Q_{[n]}$ is a quadratic functor. The polarization of $Q_{[n]}$ is the functor $B_{[n]}$ given by

$$B_{[n]}(X, X') = \varprojlim_{S \in \mathcal{F}_n} B(X(S), X'(S)).$$

Proposition 2. *The bilinear functor $B_{[n]}$ is representable. Its associated duality functor $\mathbb{D}_{[n]}$ is described by the formula*

$$(\mathbb{D}_{[n]} X)(S) = \varprojlim_{T \subseteq S} \mathbb{D}(X(T)).$$

Proof. For simplicity let us assume the existence of $\mathbb{D}_{[n]}$ and show that it is characterized by the above formula (the existence is proven in essentially the same way). We will show that for each object $C \in \mathcal{C}$, there is a canonical homotopy equivalence of spectra

$$\mathrm{Mor}_{\mathcal{C}}(C, (\mathbb{D}_{[n]} X)(S)) \simeq \varprojlim_{T \subseteq S} \mathrm{Mor}_{\mathcal{C}}(C, \mathbb{D}(X(T))).$$

Let $Y : \mathcal{F}_n^{op} \to \mathcal{C}$ be given by the formula

$$Y(T) = \begin{cases} C & \text{if } T \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(C, (\mathbb{D}_{[n]}X)(S)) &\simeq \mathrm{Mor}_{\mathcal{C}_{[n]}}(Y, \mathbb{D}_{[n]}X) \\ &\simeq B_{[n]}(Y, X) \\ &\simeq \varprojlim_{T} B(Y(T), X(T)) \\ &\simeq \varprojlim_{T \subseteq S} B(C, X(T)) \\ &\simeq \varprojlim_{T \subseteq S} \mathrm{Mor}_{\mathcal{C}}(C, \mathbb{D}X(T)) \\ &\simeq \mathrm{Mor}_{\mathcal{C}}(C, \varprojlim_{T \subseteq S} X(T)). \end{aligned}$$

□

Proposition 3. *The bilinear functor $B_{[n]}$ is nondegenerate.*

Proof. We must show that the canonical map $\mathrm{id} \to \mathbb{D}_n^2$ is an equivalence from $\mathcal{C}_{[n]}$ to itself. Fix an object $X \in \mathcal{C}_{[n]}$. We compute

$$\begin{aligned} (\mathbb{D}_{[n]}^2 X)(S) &\simeq \varprojlim_{T \subseteq S} \mathbb{D}((\mathbb{D}_{[n]}X)(T)) \\ &\simeq \mathbb{D} \varinjlim_{T \subseteq S} (\mathbb{D}_{[n]}X)(T) \\ &\simeq \mathbb{D} \varinjlim_{T \subseteq S} \varprojlim_{U \subseteq T} \mathbb{D}X(U) \\ &\simeq \mathbb{D}^2 \varprojlim_{T \subseteq S} \varinjlim_{U \subseteq T} X(U) \\ &\simeq \varprojlim_{T \subseteq S} \varinjlim_{U \subseteq T} X(U). \end{aligned}$$

We wish to show that the canonical map

$$X(S) \to \varprojlim_{T \subseteq S} \varinjlim_{U \subseteq T} X(U)$$

is an equivalence. Let $\mathcal{P}$ be the collection of all subsets of S. We define a cubical diagram $Y : \mathcal{P} \to \mathcal{C}$ by the formula

$$Y(T) = \begin{cases} X(S) & \text{if } T = \emptyset \\ \varinjlim_{\emptyset \neq U \subseteq T} X(U) & \text{otherwise.} \end{cases}$$

We wish to show that Y is a homotopy limit cube in $\mathcal{C}$. Because $\mathcal{C}$ is stable, this is equivalent to the condition that Y is a homotopy colimit cube, which follows from unwinding the definitions. For example, when S has two elements $\{s\}$ and $\{t\}$, then Y is the diagram

$$\begin{array}{ccc} X(S) & \longrightarrow & X(\{s\}) \\ \downarrow & & \downarrow \\ X(\{t\}) & \longrightarrow & X(\{s\}) \coprod_{X(S)} X(\{t\}). \end{array}$$

$\square$

Example 4. Let $n = 1$. Then an object X of $\mathcal{C}_{[1]}$ consists of a diagram

$$X(\{0\}) \leftarrow X([1]) \rightarrow X(\{1\})$$

in $\mathcal{C}$. The spectrum $Q_{[2]}(X)$ is given by the homotopy fiber product

$$Q(X(\{0\})) \times_{Q(X([1]))} Q(X(\{1\})).$$

In particular, we can identify a point of $\Omega^\infty Q_{[2]}(X)$ with a point $q_0 \in \Omega^\infty Q(X(\{0\}))$, a point $q_1 \in \Omega^\infty Q(X(\{1\}))$, and a path joining their images in $\Omega^\infty Q(X([1]))$. Such a point determines an equivalence $X \rightarrow \mathbb{D}_2 X$ if and only if the following three conditions are satisfied:

- q_0 induces an equivalence $v_0 : X(\{0\}) \rightarrow \mathbb{D}X(\{0\})$
- q_1 induces an equivalence $X(\{1\}) \rightarrow \mathbb{D}X(\{1\})$
- The induced map $v : X([1]) \rightarrow \mathbb{D}X(\{0\}) \times_{\mathbb{D}X([1])} \mathbb{D}X(\{1\})$ is an equivalence.

Note that v fits into a commutative diagram of fiber sequences

$$\begin{array}{ccccc} \mathrm{fib}(X([1]) \rightarrow X(\{0\})) & \longrightarrow & X([1]) & \longrightarrow & X(\{0\}) \\ \downarrow u & & \downarrow v & & \downarrow v_0 \\ \mathrm{fib}(\mathbb{D}X(\{1\}) \rightarrow \mathbb{D}X([1])) & \longrightarrow & \mathbb{D}X(\{0\}) \times_{\mathbb{D}X([1])} \mathbb{D}X(\{1\}) & \longrightarrow & \mathbb{D}X(\{0\}) \end{array}$$

where u is the map appearing in the previous lecture. If v_0 is an isomorphism, then v is an isomorphism if and only if u is an isomorphism.

We can summarize the situation as follows: giving a Poincare object of $\mathcal{C}_{[1]}$ is equivalent to giving a pair of Poincare objects of $\mathcal{C} = \mathcal{C}_{[0]}$, together with a cobordism between them.

The construction $[n] \mapsto \mathcal{C}_{[n]}$ is contravariantly functorial in the finite set $[n] = \{0, \ldots, n\}$. Given a map of finite sets $f : [m] \rightarrow [n]$, there is an induced functor $f^* : \mathcal{C}_{[n]} \rightarrow \mathcal{C}_{[m]}$, given by $(f^*X)(S) = X(f(S))$. Note that there is a canonical map

$$Q_{[n]}(X) = \varprojlim_{S \subseteq [n]} Q(X(S)) \rightarrow \varprojlim_{T \subseteq [m]} Q(X(f(T))) \simeq Q_{[m]}(f^*X).$$

In particular, every quadratic object (X, q) of $\mathcal{C}_{[n]}$ determines a quadratic object (f^*X, f^*q) of $\mathcal{C}_{[m]}$.

Proposition 5. *In the situation above, if (X, q) is a Poincare object of $\mathcal{C}_{[n]}$, then (f^*X, f^*q) is a Poincare object of $\mathcal{C}_{[m]}$.*

Proof. Fix a nonempty set $S \subseteq [m]$; we wish to show that q induces an isomorphism

$$(f^*X)(S) \rightarrow \varprojlim_{T \subseteq S} \mathbb{D}(f^*X)(T).$$

We can rewrite this map as a composition

$$X(f(S)) \stackrel{\phi}{\rightarrow} \varprojlim_{U \subseteq f(S)} \mathbb{D}X(U) \stackrel{\psi}{\rightarrow} \varprojlim_{T \subseteq S} \mathbb{D}X(f(T)).$$

Here the map ϕ is an isomorphism if q is nondegenerate, and ψ is an isomorphism by a cofinality argument. $\square$

For each $n \geq 0$, let $\operatorname{Poinc}(\mathcal{C}, Q)_n$ denote a classifying space for Poincare objects of $(\mathcal{C}_{[n]}, Q_{[n]})$. It follows from the preceding result that a map of finite sets $f : [m] \to [n]$ induces a map of classifying spaces $\operatorname{Poinc}(\mathcal{C}, Q)_n \to \operatorname{Poinc}(\mathcal{C}, Q)_m$. Restricting our attention to order-preserving maps f, we see that $\operatorname{Poinc}(\mathcal{C}, Q)_\bullet$ has the structure of a *simplicial space*.

Definition 6. We define $L(\mathcal{C}, Q)$ to be classifying space of the simplicial space $\operatorname{Poinc}(\mathcal{C}, Q)_\bullet$. We will refer to $L(\mathcal{C}, Q)$ as the *L-theory space* of $(\mathcal{C}, Q)$.

Remark 7. The set $\pi_0 L(\mathcal{C}, Q)$ can be identified with the quotient of $\pi_0 \operatorname{Poinc}(\mathcal{C}, Q)$ by the equivalence relation generated by the image of $\pi_0 \operatorname{Poinc}(\mathcal{C}, Q)_1$ in $\pi_0 \operatorname{Poinc}(\mathcal{C}, Q) \times \pi_0 \operatorname{Poinc}(\mathcal{C}, Q)$. Using Example 4, we see that this is exactly the relation of cobordism defined in the previous lecture. It follows that we have a canonical isomorphism

$$\pi_0 L(\mathcal{C}, Q) \simeq L_0(\mathcal{C}, Q).$$

All of the constructions of this lecture are compatible with the formation of direct sums of Poincare objects. It follows that the L-theory space $L(\mathcal{C}, Q)$ inherits a monoid structure, which is commutative and associative up to coherent homotopy: that is, $L(\mathcal{C}, Q)$ is an E_∞-space. Moreover, we saw in the last lecture that the induced monoid structure on $\pi_0 L(\mathcal{C}, Q) \simeq L_0(\mathcal{C}, Q)$ is actually an abelian group structure. In other words, $L(\mathcal{C}, Q)$ is a *grouplike E_∞-space*, and therefore an infinite loop space.

Remark 8. We will later construct a *nonconnective* delooping of $L(\mathcal{C}, Q)$.

Definition 9. For $n \geq 0$, we let $L_n(\mathcal{C}, Q)$ denote the homotopy group $\pi_n L(\mathcal{C}, Q)$. We will refer to $L_n(\mathcal{C}, Q)$ as the *nth L-group of* $(\mathcal{C}, Q)$.

We will return to the study of these higher L-groups in the next lecture.