L Groups (Lecture 5)

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Let $\mathcal{C}$ be a stable ∞-category equipped with a quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$. The polarization B of Q is a symmetric bilinear functor on $\mathcal{C}$. We will say that Q is *nondegenerate* if B is nondegenerate: that is, if there is an equivalence of ∞-categories $\mathbb{D}_Q : \mathcal{C}^{op} \to \mathcal{C}$ such that $B(X, Y) = \mathrm{Mor}_{\mathcal{C}}(X, \mathbb{D}_Q Y)$.

Assume now Q is a nondegenerate quadratic functor on a symmetric monoidal ∞-category $\mathcal{C}$. Our objective in this lecture is to define an abelian group $L_0(\mathcal{C}, Q)$, which we will call the *L-group* of the pair $(\mathcal{C}, Q)$.

Definition 1. Let Q be nondegenerate quadratic functor on a stable ∞-category $\mathcal{C}$. A *quadratic object* of $(\mathcal{C}, Q)$ is a pair (X, q), where $X \in \mathcal{C}$ and q is a point of the 0th space $\Omega^\infty Q(X)$. In this case, q determines a point in the zeroth space of $B(X, X)^{h\Sigma_2}$, hence a map $X \to \mathbb{D}_Q X$. We will say that (X, q) is a *Poincare object* if this map is invertible.

We can describe the intuition behind Definition 1 as follows: we think of Q as a functor which assigns to each object $X \in \mathcal{C}$ a "spectrum of quadratic forms on X". A quadratic object of $(\mathcal{C}, Q)$ can then be thought of as an object of $\mathcal{C}$ equipped with a some type of quadratic form (whose exact nature depends on Q), and a Poincare object of $(\mathcal{C}, Q)$ as an object of $\mathcal{C}$ equipped with a nondegenerate quadratic form.

Example 2. Here is the motivating example. Fix an integer $n \geq 0$. Let $B : \mathcal{D}^{\mathrm{perf}}(\mathbf{Z})^{op} \times \mathcal{D}^{\mathrm{perf}}(\mathbf{Z})^{op} \to \mathrm{Sp}$ be the bilinear functor given informally by the formula

$$(P_\bullet, Q_\bullet) \mapsto \mathrm{Mor}_{\mathcal{D}^{\mathrm{perf}}(\mathbf{Z})}(P_\bullet \otimes Q_\bullet, \mathbf{Z}[-n])$$

(here $\mathbf{Z}[-n]$ denotes the chain complex consisting of the single abelian group $\mathbf{Z}$, concentrated in homological degree $-n$). Then B is a symmetric bilinear functor; let $Q : \mathcal{D}^{\mathrm{perf}}(\mathbf{Z})^{op} \to \mathrm{Sp}$ be the quadratic functor given by $Q(P_\bullet) = B(P_\bullet, P_\bullet)^{h\Sigma_2}$.

Let M be a compact oriented manifold of dimension n. We can identify the singular cochain complex $C^*(M; \mathbf{Z})$ with an object of $\mathcal{D}^{\mathrm{perf}}(\mathbf{Z})$. The intersection pairing

$$C^*(M; \mathbf{Z}) \otimes C^*(M; \mathbf{Z}) \to C^*(M; \mathbf{Z}) \xrightarrow{[M]} \mathbf{Z}[-n]$$

determines a point $q_M \in \Omega^\infty Q(C^*(M; \mathbf{Z}))$. Poincare duality is equivalent to the assertion that the pair $(C^*(M; \mathbf{Z}), q_M)$ is a Poincare object of $(\mathcal{D}^{\mathrm{perf}}(\mathbf{Z}), Q)$.

Example 3. Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor Q. The space $\Omega^\infty Q(0)$ is contractible; let q denote any point of this contractible space. Then the pair $(0, q)$ is a Poincare object of $(\mathcal{C}, Q)$.

Example 4. Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor Q. Suppose we are given quadratic objects (X, q) and (X', q') of $(\mathcal{C}, Q)$. Let $q \oplus q'$ denote the image of (q, q') under the map $Q(X) \oplus Q(X') \to Q(X \oplus X')$. The pair $(X \oplus X', q \oplus q')$ is another quadratic object of $(\mathcal{C}, Q)$, which we call the *sum* of (X, q) and (X', q') and denote by $(X, q) \oplus (X', q')$. Note that if (X, q) and (X', q') are Poincare objects, then $(X \oplus X', q \oplus q')$ is also a Poincare object.

If $\mathcal{C}$ is a stable ∞-category equipped with a nondegenerate quadratic functor, then the collection of homotopy equivalence classes of Poincare objects forms a commutative monoid with respect to the addition of Example 4; the unit for this addition is the zero Poincare object given in Example 3. However, this monoid is evidently not a group: if $(X, q) \oplus (X', q') \simeq 0$, then we must have $X \simeq X' \simeq 0$. We will correct this problem by introducing a suitable equivalence relation on Poincare objects.

Definition 5. Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$, and suppose we are given Poincare objects (X, q) and (X', q'). A *cobordism* from (X, q) to (X', q') consists of the following data:

(*i*) An object $L \in \mathcal{C}$ equipped with maps $\alpha : L \to X$ and $\alpha' : L \to X'$.

(*ii*) A path p joining the images of q and q' in the space $\Omega^{\infty} Q(L)$.

Moreover, this data must satisfy the following nondegeneracy condition:

(*iii*) The diagram

$$\begin{array}{ccccc} X & \xleftarrow{\alpha} & L & \xrightarrow{\alpha'} & X' \\ \downarrow & & & & \downarrow \\ \mathbb{D}_Q(X) & \xrightarrow{\mathbb{D}_Q(\alpha)} & \mathbb{D}_Q(L) & \xleftarrow{\mathbb{D}_Q(\alpha')} & \mathbb{D}_Q(X') \end{array}$$

commutes up to a homotopy determined by the path p. It follows that the composition

$$\mathrm{fib}(\alpha) \to L \xrightarrow{\alpha'} X' \to \mathbb{D}_Q(X') \to \mathbb{D}_Q(L)$$

is canonically nullhomotopic, so we obtain a map of fibers

$$u : \mathrm{fib}(\alpha) \to \mathrm{fib}(\mathbb{D}_Q(\alpha'))$$

or, more informally, a map $u : \Omega X/L \to \mathbb{D}_Q(X'/L)$. We require that u is invertible.

We will say that a pair of Poincare objects (X, q) and (X', q') are *cobordant* if there is a cobordism from (X, q) to (X', q').

Example 6. Let M and M' be compact oriented n-manifolds, and let $(C^*(M; \mathbf{Z}), q_M)$, $(C^*(M'; \mathbf{Z}), q_{M'})$ be the Poincare objects of $\mathcal{D}^{\mathrm{perf}}(\mathbf{Z})$ described in Example 2. Suppose that B is an (oriented) bordism from M to M', and let $L = C^*(B; \mathbf{Z})$ be the singular cochain complex of B. Then we have restriction maps $\alpha : L \to C^*(M; \mathbf{Z})$ and $\alpha' : L \to C^*(M'; \mathbf{Z})$. Moreover, the images of q_M and $q_{M'}$ in $\Omega^{\infty} Q(L)$ are joined by a canonical path, because the difference of fundamental homology classes $[M] - [M']$ in B is given as the boundary of the fundamental homology class of B. This path exhibits L as a cobordism from the Poincare object $(C^*(M; \mathbf{Z}), q_M)$ to the Poincare object $(C^*(M'; \mathbf{Z}), q_{M'})$: unwinding the definitions, this amounts to verifying that cap product with the fundamental class of B induces isomorphisms

$$\mathrm{H}^m(B, M; \mathbf{Z}) \to \mathrm{H}_{n+1-m}(B, M'; \mathbf{Z})$$

(which is a form of Poincare duality for manifolds with boundary).

Example 7. An important special case of Definition 5 occurs when (X, q) is the zero Poincare object. In this case, a cobordism from (X, q) to (X', q') is given by a map $\beta : L \to X'$ and a nullhomotopy of the image of q' in $Q(L)$, satisfying a nondegeneracy condition which requires that the induced map $u : L \to \mathrm{fib}(\mathbb{D}_Q(\beta)) \simeq \mathbb{D}_Q \,\mathrm{cofib}(\beta) = \mathbb{D}_Q X'/L$ is an equivalence. In this case, we will say that L is a *Lagrangian* in (X', q) (this terminology is slightly abusive: the condition of being a Lagrangian depends not only on L, but also on the map β and the choice of nullhomotopy).

Example 8. In the situation of Definition 5, suppose that (X, q) and (X', q') are both zero Poincare objects. Then a cobordism from (X, q) to (X', q') can be identified with an object $L \in \mathcal{C}$ together with a point $p \in \Omega^{\infty+1}Q(L)$ which induces an equivalence $L \to \Omega\mathbb{D}_Q(L)$. In other words, a cobordism from (X, q) to (X', q') can be identified with a Poincare object of $\mathcal{C}$ with respect to the shifted quadratic functor ΩQ.

Proposition 9. *Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor Q. The relation of cobordism is an equivalence relation on the collection of Poincare objects of $(\mathcal{C}, Q)$.*

Proof. We first show that cobordism is reflexive. Let (X, q) be a Poincare object of $(\mathcal{C}, Q)$. Take $L = X$ and let $\alpha : L \to X$ and $\alpha' : L \to X$ be the identity maps. Let p be the constant path between the images of q in $\Omega^\infty Q(L)$. Then (L, α, α', p) is a cobordism from (X, q) to itself.

We next show that cobordism is symmetric. Let (X, q) and (X', q') be Poincare objects of $(\mathcal{C}, Q)$, and suppose we are given a diagram

$$X \overset{\alpha}{\leftarrow} L \overset{\alpha'}{\rightarrow} X'$$

in $\mathcal{C}$ and a path joining the images of q and q' in $\Omega^\infty Q(L)$. We claim that if this data is a cobordism from (X, q) to (X', q'), then it is also a cobordism from (X', q') to (X, q). Condition (iii) of Definition 5 guarantees that the canonical map

$$u : \mathrm{fib}(\alpha) \to \mathrm{fib}(\mathbb{D}_Q(\alpha')) \simeq \mathbb{D}_Q \,\mathrm{cofib}(\alpha')$$

is an equivalence. We wish to show that the canonical map

$$v : \mathrm{fib}(\alpha') \to \mathrm{fib}(\mathbb{D}_Q(\alpha)) \simeq \mathbb{D}_Q \,\mathrm{cofib}(\alpha)$$

is also an equivalence, or equivalently that

$$\Sigma(v) : \mathrm{cofib}(\alpha') \to \mathbb{D}_Q \,\mathrm{fib}(\alpha)$$

is an equivalence. For this, one shows that $\Sigma(v)$ agrees with $\mathbb{D}_Q(u)$ up to a sign.

We now show that cobordism is transitive. Suppose we are given a triple of Poincare objects (X, q), (X', q'), and (X'', q''), together with a diagram

$$X \overset{\alpha}{\leftarrow} L \overset{\alpha'}{\rightarrow} X' \overset{\beta}{\leftarrow} L' \overset{\beta'}{\rightarrow} X'',$$

a path p joining the image of q and q' in $\Omega^\infty Q(L)$, and a path p' joining the images of q' and q'' in $\Omega^\infty Q(L')$. Let S denote the fiber product $L \times_{X'} L'$. We have evident maps $X \overset{\gamma}{\leftarrow} S \overset{\gamma'}{\rightarrow} X''$ so that the concatentation of p and p' determines a path between the images of q and q'' in the space $\Omega^\infty Q(S)$. We claim that this path exhibits S as a cobordism from (X, q) to (X'', q''). To prove this, we must show that the induced map $u : \mathrm{fib}(\gamma) \to \mathrm{fib}(\mathbb{D}(\gamma'))$ is invertible. It now suffices to observe that this map fits into a diagram of fiber sequences

$$\begin{array}{ccccc}
\mathrm{fib}(\beta) & \longrightarrow & \mathrm{fib}(\gamma) & \longrightarrow & \mathrm{fib}(\alpha) \\
\downarrow & & \downarrow u & & \downarrow \\
\mathrm{fib}(\mathbb{D}_Q(\beta')) & \longrightarrow & \mathrm{fib}(\mathbb{D}_Q(\gamma')) & \longrightarrow & \mathrm{fib}(\mathbb{D}_Q(\alpha'))
\end{array}$$

where the left and right vertical maps are invertible by virtue of our assumptions that we have cobordisms from (X', q') to (X'', q'') and (X, q) to (X', q'), respectively. □

Definition 10. Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor Q. We let $L_0(\mathcal{C}, Q)$ denote the set of cobordism classes of Poincare objects of $(\mathcal{C}, Q)$.

The direct sum operation on Poincare objects descends to give an addition on the set $L_0(\mathcal{C}, Q)$ (since there is a corresponding direct sum operation on cobordisms themselves), making $L_0(\mathcal{C}, Q)$ into a commutative monoid. In fact, $L_0(\mathcal{C}, Q)$ is an abelian group. Suppose that (X, q) is a Poincare object of $(\mathcal{C}, Q)$. Since $\pi_0 Q(X)$ is an abelian group, we can choose a point $-q \in \Omega^\infty Q(X)$ which is inverse to q up to homotopy. Note that the pair $(X, -q)$ is also a Poincare object of $(\mathcal{C}, Q)$, which is determined up to (noncanonical) homotopy equivalence by (X, q). We claim that this Poincare object is an inverse to (X, q) in $L_0(\mathcal{C}, Q)$:

Proposition 11. *In the above situation, we have* $(X, q) \oplus (X, -q) = 0$ *in* $L_0(\mathcal{C}, Q)$. *That is, there is a cobordism from* $(X \oplus X, q \oplus -q)$ *to the zero Poincare object.*

Proof. By Example 7, we must show that there exists a Lagrangian $\beta : L \to X \oplus X$. For this, we take $L = X$ and β to be the diagonal map, and choose any path from the sum $(q + -q) \in \Omega^\infty Q(X)$ to the base point. The requisite nondegeneracy condition follows from our assumption that q induces an equivalence $X \to \mathbb{D}_Q X$. $\square$

By virtue of the above result, we are now justified in referring to $L_0(\mathcal{C}, Q)$ as the 0*th* L*-group* of the pair $(\mathcal{C}, Q)$.

Remark 12. Let M be a compact oriented manifold of dimension n, and let $(C^*(M; \mathbf{Z}), q_M)$ as in Example 6. Then $(C^*(M; \mathbf{Z}), q_M)$ determines an element of $L_0(\mathcal{D}^{\mathrm{perf}}(\mathbf{Z}), Q)$, and this element is an incarnation of the *signature* of the manifold M. (In fact, when n is divisible by 4 one can show that $L_0(\mathcal{D}^{\mathrm{perf}}(\mathbf{Z}), Q)$ is isomorphic to $\mathbf{Z}$ and this invariant is precisely the signature). We will ultimately need a more refined version of the signature invariant in order to describe the surgery classification of manifolds. However, this more refined invariant will have the same basic flavor: it will live in a group $L_0(\mathcal{C}, Q)$ for some quadratic functor on a stable ∞-category $\mathcal{C}$, and the invariant associated to M will be some avatar of the stable homotopy type of M, equipped with its intersection product.