Quadratic Functors (Lecture 4)

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In this lecture, we will introduce the notion of a *quadratic functor* Q on a stable ∞-category $\mathcal{C}$, and define the L-group $L_0(\mathcal{C}, Q)$. We begin with a short review of the classical theory of quadratic forms.

Definition 1. Let M and A be abelian groups. An *A-valued bilinear form* on M is a map

$$b : M \times M \to A$$

such that, for each $x \in M$, the maps $y \mapsto b(x, y)$ and $y \mapsto b(y, x)$ are abelian group homomorphisms from M into A. We will say that b is *symmetric* if $b(x, y) = b(y, x)$.

An *inhomogeneous A-valued quadratic form on M* is a map $q : M \to A$ such that $q(0) = 0$ and the function $b(x, y) = q(x + y) - q(x) - q(y)$ is a bilinear form. We will say that q is a *quadratic form* if, in addition, we have $q(nx) = n^2 q(x)$ for every integer n and every $x \in M$.

The theory of quadratic forms and bilinear forms are closely connected. If q is an inhomogeneous quadratic form on an abelian group M, then the function $b(x, y) = q(x+y) - q(x) - q(y)$ is a symmetric bilinear form. If multiplication by 2 is invertible on A, we can almost recover q from the bilinear form b: namely, we have $q(x) = \frac{1}{2} b(x, x) + l(x)$ for some group homomomorphism $l : M \to A$. In particular, the construction $b \mapsto \frac{1}{2} b(x, x)$ determines a bijective correspondence between symmetric bilinear forms and quadratic forms (whenever multiplication by 2 is invertible on A).

Our next goal is to *categorify* some of these ideas: that is, to make sense of the algebraic structures described above when the notion of module is replaced by some sort of category (in our case, stable ∞-categories). Let us begin by drawing up a table of analogies:

Classical Story	Categorified Story
abelian group	stable ∞-category
$\mathbf{Z}$	∞-category Sp of spectra
abelian group homomorphism	exact functor
(symmetric) bilinear form	(symmetric) bilinear functor
inhomogeneous quadratic form	quadratic functor

We now introduce some of the relevant definitions.

Definition 2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between stable ∞-categories. We say that F is *exact* if it carries zero objects to zero objects and fiber sequences to fiber sequences.

Let Sp denote the ∞-category of spectra.

Definition 3. Let $\mathcal{C}$ be a stable ∞-category. A *bilinear functor* on $\mathcal{C}$ is a functor

$$B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$$

with the following property: for every object $C \in \mathcal{C}$, the functors

$$D \mapsto B(C, D) \qquad D \mapsto B(D, C)$$

are exact functors from $\mathcal{C}^{op}$ to Sp.

The collection of bilinear functors on $\mathcal{C}$ is evidently acted on by the symmetric group Σ_2 on two letters (by permuting the arguments). A *symmetric bilinear functor* is a homotopy fixed point for this action.

Let $\mathcal{C}$ be a stable ∞-category containing an object X. For every object Y, the sequence of mapping spaces $\{\mathrm{Map}_{\mathcal{C}}(Y, \Sigma^n X)\}_{n \geq 0}$ constitutes a spectrum (that is, each is homotopy equivalent to the loop space on the next). We will denote this spectrum by $\mathrm{Mor}_{\mathcal{C}}(Y, X)$. The construction $Y \mapsto \mathrm{Mor}_{\mathcal{C}}(Y, X)$ determines an (exact) functor from $\mathcal{C}^{op}$ to Sp. We will say that a functor $F : \mathcal{C}^{op} \to \mathrm{Sp}$ is *representable* if it arises in this way.

Suppose that $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$ is a symmetric bilinear functor. We will say that B is *representable* if, for all $X \in \mathcal{C}$, the functor $Y \mapsto B(X, Y)$ is representable. In this case, we write $B(X, Y) = \mathrm{Mor}_{\mathcal{C}}(Y, \mathbb{D}X)$ for some object $\mathbb{D}X$ in $\mathcal{C}$, which is determined up to contractible ambiguity. The construction $X \mapsto \mathbb{D}X$ determines a functor from $\mathcal{C}$ to $\mathcal{C}^{op}$. For each $X \in \mathcal{C}$, the identity map $\mathrm{id}_{\mathbb{D}X}$ determines point in the zeroth space of $B(X, \mathbb{D}X) \simeq B(\mathbb{D}X, X)$, and therefore a morphism $e_X : X \to \mathbb{D}^2 X$. We will say that B is *nondegenerate* if it is representable and the canonical map e_X is an equivalence for every $X \in \mathcal{C}$.

Example 4. Let $\mathcal{C}$ be the ∞-category of spectra, and let $\wedge$ denote the smash product functor. The functor $B(X, Y) = \mathrm{Mor}_{\mathrm{Sp}}(X \wedge Y, S)$ determines a symmetric bilinear functor on $\mathcal{C}$. This symmetric bilinear functor is representable, and the corresponding functor $\mathbb{D} : \mathcal{C} \to \mathcal{C}$ is *Spanier-Whitehead duality.* If we restrict our attention to the full subcategory of $\mathcal{C}$ spanned by the *finite* spectra, then B becomes nondegenerate.

Definition 5. Let $\mathcal{C}$ be a stable ∞-category. We say that a functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ is *reduced* if Q carries zero objects to zero objects. In this case, Q carries also carries zero morphisms to zero morphisms.

Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a reduced functor. If $X, Y \in \mathcal{C}$, we obtain maps

$$Q(X) \oplus Q(Y) \to Q(X \oplus Y) \to Q(X) \oplus Q(Y)$$

where the composition is given by applying Q to the matrix

$$\begin{bmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{bmatrix}$$

If Q is reduced, this map is the identity so that $Q(X) \oplus Q(Y)$ is a summand of $Q(X \oplus Y)$; that is, we have a direct sum decomposition $Q(X \oplus Y) \simeq Q(X) \oplus Q(Y) \oplus B(X, Y)$, for some functor $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$. We will refer to B as the *polarization of* Q. Note that B is manifestly symmetric in its arguments.

Suppose we are given a reduced functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ with polarization B. For every object $X \in \mathcal{C}$, the codiagonal map $X \oplus X \to X$ induces a map $Q(X) \to Q(X \oplus X)$. Projecting onto the component $B(X, X)$, we obtain a map $Q(X) \to B(X, X)$. This construction is evidently Σ_2-invariant, and gives a map $Q(X) \to B(X, X)^{h\Sigma_2}$ (here $B(X, X)^{h\Sigma_2}$ denotes the homotopy fixed point spectrum for the action of Σ_2 on $B(X, X)$).

Definition 6. Let $\mathcal{C}$ be a stable ∞-category and let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a functor. We will say that Q is *quadratic* if the following conditions are satisfied:

(1) The functor Q is reduced.

(2) The polarization B of Q is bilinear.

(3) The functor $X \mapsto \mathrm{fib}(Q(X) \to B(X,X)^{h\Sigma_2})$ is exact.

Example 7. Let $\mathcal{C}$ be a stable ∞-category and let B be a symmetric bilinear functor on $\mathcal{C}$. Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be given by the formula $Q(X) = B(X,X)^{h\Sigma_2}$, and let B' be the polarization of Q. A simple calculation gives

$$B'(X,Y) = (B(X,Y) \oplus B(Y,X))^{h\Sigma_2} \simeq B(X,Y),$$

and that the canonical map $Q(X) \to B'(X,X)^{h\Sigma_2}$ is an equivalence. Consequently, Q is a quadratic functor.

Lemma 8. *Let $\mathcal{C}$ be a stable ∞-category and let $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$ be a bilinear functor. Define $F : \mathcal{C}^{op} \to \mathrm{Sp}$ by the formula*

$$F(X) = B(X,X)^{t\Sigma_2};$$

here the superscript indicates Tate cohomology. Then the functor F is exact.

Proof. Suppose we are given a fiber sequence $X' \to X \to X''$ in $\mathcal{C}$. We then obtain a diagram

$$\begin{array}{ccccc}
B(X'',X'') & \longrightarrow & B(X'',X) & \longrightarrow & B(X'',X') \\
\downarrow & & \downarrow & & \downarrow \\
B(X,X'') & \longrightarrow & B(X,X) & \longrightarrow & B(X,X') \\
\downarrow & & \downarrow & & \downarrow \\
B(X',X'') & \longrightarrow & B(X',X) & \longrightarrow & B(X',X')
\end{array}$$

in which the rows and columns are fiber sequences. It follows that we have a fiber sequence

$$B(X'',X'') \to B(X,X) \to B(X,X') \times_{B(X',X')} B(X',X).$$

We can rewrite the third term as

$$(B(X,X') \times B(X',X)) \times_{B(X',X')^2} B(X',X'),$$

so have a fiber sequence

$$B(X'',X'') \to B(X,X) \to (B(X,X') \times B(X',X)) \times_{B(X',X')^2} B(X',X').$$

Passing to Tate cohomology (and using the fact that the Tate cohomology vanishes on an induced representation) we get a fiber sequence

$$B(X'',X'')^{t\Sigma_2} \to B(X,X)^{t\Sigma_2} \to B(X',X')^{t\Sigma_2}.$$

□

Example 9. Let $\mathcal{C}$ be a stable ∞-category and let B be a symmetric bilinear functor on $\mathcal{C}$. Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be given by the formula $Q(X) = B(X,X)_{h\Sigma_2}$, the *homotopy coinvariants* for the action of Σ_2 on $B(X,X)$, and let B' be the polarization of Q. A simple calculation gives $B'(X,Y) = (B(X,Y) \oplus B(Y,X))_{h\Sigma_2} \simeq B(X,Y)$. Moreover, the canonical map

$$Q(X) \to B'(X,Y)^{h\Sigma_2}$$

can be identified with the *norm map*

$$B(X,X)_{h\Sigma_2} \to B(X,X)^{h\Sigma_2}.$$

The cofiber of this map is the Tate cohomology spectrum $B(X,X)^{t\Sigma_2}$. The functor $X \mapsto B(X,X)^{t\Sigma_2}$ is an exact functor of X, so that Q is a quadratic functor.

Remark 10. Let $\mathcal{C}$ be a stable ∞-category and let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a reduced functor with polarization B. Using the diagonal map $X \to X \oplus X$ instead of the codiagonal in the preceding discussion, we obtain a canonical map $B(X,X)_{h\Sigma_2} \to Q(X)$. The composition

$$B(X,X)_{h\Sigma_2} \to Q(X) \to B(X,X)^{h\Sigma_2}$$

is given by the norm map (averaging with respect to the action of Σ_2). If the homotopy groups of the spectrum $B(X,X)$ are uniquely 2-divisible, then this norm map is a homotopy equivalence of spectra. It follows in this case that we obtain a direct sum decomposition

$$Q(X) \simeq B(X,X)^{h\Sigma_2} \oplus L(X) \simeq B(X,X)_{h\Sigma_2} \oplus L(X)$$

for some reduced functor $L : \mathcal{C}^{op} \to \mathrm{Sp}$ with trivial polarization. Then Q is quadratic if and only if B is bilinear and L is exact. We can informally summarize the situation as follows: if we work in the setting where 2 is invertible (for example, if multiplication by 2 induces an isomorphism from each object of $\mathcal{C}$ to itself), then every quadratic functor on $\mathcal{C}$ decomposes uniquely as the sum of an exact functor and a functor of the form $B(X,X)^{h\Sigma_2}$, where B is a symmetric bilinear functor on $\mathcal{C}$.

Remark 11. Our definition of quadratic functor is a special case of a much more general notion which arises in Goodwillie's *calculus of functors*.

Remark 12. Let $\mathcal{C}$ be a stable ∞-category. Suppose we are given a fiber sequence

$$Q_0(X) \to Q(X) \to B(X,X)^{h\Sigma_2}$$

for some symmetric bilinear functor $B : \mathcal{C}^{op} \times \mathcal{C}^{op} \to \mathrm{Sp}$. If Q_0 is exact, then a simple calculation shows that the polarizaton of Q is given by

$$F(X,Y) = (B(X,Y) \oplus B(Y,X))^{h\Sigma_2} \simeq B(X,Y),$$

so that Q is quadratic. In other words, a functor Q is quadratic if and only if is arises an extension of $B(X,X)^{h\Sigma_2}$ by an exact functor, for some symmetric bilinear functor B.