The Total Surgery Obstruction (Lecture 37)

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Let X be a finite polyhedron, ζ_X a spherical fibration on X, and R and A_∞-ring with involution. Recall that we have a homotopy pullback diagram of spectra

$$\begin{array}{ccc} \mathbb{L}^q(X, \zeta_X, R) & \longrightarrow & \mathbb{L}^s(X, \zeta_X, R) \\ \downarrow & & \downarrow \\ \mathbb{L}^{vq}(X, \zeta_X, R) & \longrightarrow & \mathbb{L}^{vs}(X, \zeta_X, R), \end{array}$$

where the vertical maps are given by assembly. From now on, we will fix R to be the ring $\mathbf{Z}$ of integers, and omit it from the notation (we could just as well take R to be the sphere spectrum). Let $\widehat{\mathbb{L}}(X, \zeta_X)$ denote the cofibers of the horizontal maps, so that we have a diagram

$$\begin{array}{ccccc} \mathbb{L}^q(X, \zeta_X) & \longrightarrow & \mathbb{L}^s(X, \zeta_X) & \longrightarrow & \widehat{\mathbb{L}}(X, \zeta_X) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}^{vq}(X, \zeta_X) & \longrightarrow & \mathbb{L}^{vs}(X, \zeta_X) & \longrightarrow & \widehat{\mathbb{L}}(X, \zeta_X). \end{array}$$

When the spherical fibration ζ_X is trivial; we will omit it from the notation. If, in addition, X is a point, then the vertical maps are homotopy equivalences and we have a single fiber sequence

$$\mathbb{L}^q(\mathbf{Z}) \to \mathbb{L}^s(\mathbf{Z}) \to \widehat{\mathbb{L}}.$$

The upper row of the diagram above consists of functors which are excisive in X. We may therefore write $\widehat{\mathbb{L}}(X, \zeta_X) = C_*(X, \widehat{\mathbb{L}}(\zeta_X))$, where $\widehat{\mathbb{L}}(\zeta_X)$ is the local system on X which assigns to each point $x \in X$ the spectrum $\widehat{\mathbb{L}}(\zeta_X(x))$.

Suppose we are given a map of spaces $i : \partial X \to X$. We let $\mathbb{L}^{vs}(X, \partial X, \zeta_X)$ denote the cofiber of the map $\mathbb{L}^{vs}(\partial X, \zeta_X | \partial X) \to \mathbb{L}^{vs}(X, \zeta_X)$, and define $\mathbb{L}^{vq}(X, \partial X, \zeta_X)$ and $\widehat{\mathbb{L}}(X, \partial X, \zeta_X)$ similarly. By functoriality, we obtain vertical maps fitting into a commutative diagram of spectra

$$\begin{array}{ccccc} \mathbb{L}^q(\mathbf{Z}) \wedge (X/\partial X) & \longrightarrow & \mathbb{L}^s(\mathbf{Z}) \wedge (X/\partial X) & \longrightarrow & \widehat{\mathbb{L}} \wedge (X/\partial X) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}^{vq}(X, \partial X) & \longrightarrow & \mathbb{L}^{vs}(X, \partial X) & \longrightarrow & \widehat{\mathbb{L}}(X, \partial X). \end{array}$$

The right vertical map is a homotopy equivalence (by excision). If X and ∂X have the same fundamental groupoid, then the π-π theorem implies that $\mathbb{L}^{vq}(X, \partial X)$ vanishes. The above diagram then gives a canonical homotopy equivalence

$$\widehat{\mathbb{L}} \wedge (X/\partial X) \simeq \mathbb{L}^{vs}(X, \partial X).$$

In particular, we can take $X = D^n$ and $\partial X = S^{n-1}$ for $n \geq 3$, to obtain a homotopy equivalence

$$\widehat{\mathbb{L}} \simeq \Omega^n \mathbb{L}^{vs}(D^n, S^{n-1}).$$

Recall that the symmetric L-theory spectrum $\mathbb{L}^s(\mathbf{Z})$ is an E_∞-ring spectrum, with multiplication induced by the tensor product of chain complexes. If we have pairs of spaces $(X, \partial X)$ and $(Y, \partial Y)$, there is a similar multiplicaiton

$$\mathbb{L}^{vs}(X, \partial X) \wedge \mathbb{L}^{vs}(Y, \partial Y) \to \mathbb{L}^{vs}(X \times Y, \partial(X \times Y)),$$

where $\partial(X \times Y)$ denotes the homotopy pushout $(\partial X \times Y) \coprod_{\partial X \times \partial Y} (X \times \partial Y)$. Taking X and Y to be disks of dimension ≥ 3, this gives a multiplication

$$\widehat{\mathbb{L}} \wedge \widehat{\mathbb{L}} \to \widehat{\mathbb{L}}.$$

Using more elaborate reasoning along the same lines, we see that $\widehat{\mathbb{L}}$ also has the structure of an E_∞-ring spectrum, and that the canonical map $\mathbb{L}^s(\mathbf{Z}) \to \widehat{\mathbb{L}}$ can be promoted to a map of E_∞-ring spectra.

Now suppose that $(X, \partial X)$ is a Poincare pair with Spivak fibration ζ_X, so that the visible symmetric signature $\sigma_X^{vs} \in \Omega^\infty \mathbb{L}^s(X, \partial_X, \zeta_X)$ is defined. Let $\widehat{\sigma}_X$ denote the image of σ_X^{vs} in $\Omega^\infty \widehat{\mathbb{L}}(X, \partial X, \zeta_X)$.

Let us now suppose that $\partial X = S^{n-1}$ is a sphere, and that $X = D^n$ is the cone on ∂X. Then the Spivak bundle of X is canonically equivalent to the constant sheaf whose value is an invertible specturm E, given by the *inverse* of $\Sigma^\infty(X/\partial X)$. Then $\widehat{\mathbb{L}}(X, \partial X, \zeta_X)$ can be identified with the spectrum $E^{-1} \wedge \widehat{\mathbb{L}}(*, E)$. We can then identify $\widehat{\sigma}_X$ with a map $\widehat{\mathbb{L}}$-modules $\widehat{\phi} : E \wedge \widehat{\mathbb{L}} \to \widehat{\mathbb{L}}(E)$, which is a homotopy equivalence. In Lecture 23, we discussed an analogous homotopy equivalence

$$\phi : E \wedge \mathbb{L}^s(\mathbf{Z}) \simeq \mathbb{L}^s(*, E).$$

These maps fit into a commutative diagram

$$\begin{array}{ccc} E \wedge \mathbb{L}^s(\mathbf{Z}) & \xrightarrow{\phi} & \mathbb{L}^s(*, E) \\ \downarrow & & \downarrow \\ E \wedge \widehat{\mathbb{L}} & \xrightarrow{\widehat{\phi}} & \widehat{\mathbb{L}}(*, E). \end{array}$$

However, there is an important difference: to write down the map ϕ, we needed to realize the spectrum $\mathbb{L}^s(X, \partial X, \underline{E})$ in terms of constructible sheaves on X; the map ϕ itself was given by choosing the constant sheaf on X, which is Verdier-self dual (up to a twist). Consequently, ϕ is functorial with respect to PL homeomorphisms of $X = D^n$. However, the construction of $\widehat{\phi}$ depends only on the realization of $(X, \partial X)$ as a Poincare pair, and is therefore functorial with respect to all homotopy equivalences of $\partial X = S^{n-1}$.

Elaborating on the above construction, we obtain the following:

Proposition 1. *Let E be any invertible spectrum. Then there is a canonical homotopy equivalence*

$$E \wedge \widehat{\mathbb{L}} \simeq \widehat{\mathbb{L}}(*, E).$$

Now suppose we are given a point $\eta \in G/PL = \mathrm{fib}(\mathbf{Z} \times \mathrm{BPL} \to \mathrm{Pic}(S))$. Then η classifies a stable PL bundle ζ over a point, together with a trivialization of the underlying invertible spectrum E. Using ζ, we can write down a homotopy equivalence of $\mathbb{L}^s$-modules

$$\phi_\zeta : E \wedge \mathbb{L}^s(\mathbf{Z}) \to \mathbb{L}^s(*, E).$$

Since E is trivial, we can think of ϕ_ζ as an invertible map from $\mathbb{L}^s(\mathbf{Z})$ to itself. This construction gives a map

$$G/PL \to \mathrm{GL}_1(\mathbb{L}^s(\mathbf{Z}))$$

Since the automorphism of $\widehat{\mathbb{L}}$ determined by ϕ_ζ depends only on the underlying spherical fibration of ζ, this automorphism is trivial. Consequently, the composite map

$$G/PL \to \mathrm{GL}_1(\mathbb{L}^s(\mathbf{Z})) \to \mathrm{GL}_1(\widehat{\mathbb{L}})$$

is canonically nullhomotopic. We therefore obtain a map $G/PL \to F$, where F denotes the homotopy fiber of the map $\mathrm{GL}_1(\mathbb{L}^s) \to \mathrm{GL}_1(\widehat{\mathbb{L}})$. Note that F is the identity component of the space

$$L^q(\mathbf{Z}) = \Omega^\infty \operatorname{fib}(\mathbb{L}^s \to \widehat{\mathbb{L}}).$$

The above construction recovers the map $\theta : G/PL \to L^q(\mathbf{Z})$ described in the previous lecture. Recall that this map is *almost* a homotopy equivalence: we have a fiber sequence $K(\mathbf{Z}/2\mathbf{Z}, 3) \to G/PL \to L^q(\mathbf{Z})$.

Now let X be any finite polyhedron and ζ_X any spherical fibration on X. Using excision and Proposition 1, we obtain homotopy equivalences

$$\widehat{\mathbb{L}}(X, \zeta_X) \simeq C_*(X; \widehat{\mathbb{L}}(\zeta_X)) \simeq C_*(X; \zeta_X \wedge \widehat{\mathbb{L}}).$$

Here $\widehat{\mathbb{L}}(\zeta_X)$ denotes the local system on X which assigns to each point $x \in X$ the spectrum $\widehat{\mathbb{L}}(\{x\}, \zeta_X(x))$. In the special case where X is a Poincare space and ζ_X is its Spivak bundle, we obtain

$$\widehat{\mathbb{L}}(X, \zeta_X) \simeq C^*(X; \underline{\widehat{\mathbb{L}}}).$$

Under this homotopy equivalence, the point $\widehat{\sigma}_X \in \Omega^\infty \widehat{\mathbb{L}}(X, \zeta_X)$ corresponds to the global section of $\underline{\widehat{\mathbb{L}}}$ given by the unit of $\widehat{\mathbb{L}}$.

A similar calculation gives $\mathbb{L}^s(X, \zeta_X) \simeq C^*(X; \zeta_X^{-1} \wedge \mathbb{L}^s(\zeta_X))$, where $\mathbb{L}^s(\zeta_X)$ denotes the local system on X which assigns to each point $x \in X$ the spectrum $\mathbb{L}^s(\{x\}, \zeta_X(x))$. In other words, we can identify $\mathbb{L}^s(X, \zeta_X)$ with the $\mathrm{Mor}(\zeta_X \wedge \mathbb{L}^s(\mathbf{Z}), \mathbb{L}^s(\zeta_X))$ in the ∞-category of local systems of $\mathbb{L}^s(\mathbf{Z})$-modules on X. Let $\mathbb{L}^s(X, \zeta_X)^\times$ denote the subspace of $\Omega^\infty \mathbb{L}^s(X, \zeta_X)$ corresponding to *isomorphisms* $\zeta_X \wedge \mathbb{L}^s(\mathbf{Z}) \to \mathbb{L}^s(\zeta_X)$. We have a map

$$\mathbb{L}^s(X, \zeta_X)^\times \to \Omega^\infty \widehat{\mathbb{L}}(X, \zeta_X).$$

The homotopy fiber of this map over $\widehat{\sigma}_X$ can be identified with the space of sections of a fibration $X' \to X$, having fiber $F = \operatorname{fib}(\mathrm{GL}_1(\mathbb{L}^s(\mathbf{Z})) \to \mathrm{GL}_1(\widehat{\mathbb{L}}))$.

Any stable PL structure on the bundle ζ_X gives an isomorphism of local systems of $\mathbb{L}^s(\mathbf{Z})$-modules $\zeta_X \wedge \mathbb{L}^s(\mathbf{Z}) \to \mathbb{L}^s(\zeta_X)$, corresponding to a point of $\mathbb{L}^s(X, \zeta_X)^\times \times_{\Omega^\infty \widehat{\mathbb{L}}(X, \zeta_X)} \{\widehat{\sigma_X}\}$. The collection of such PL structures is classified by the space of sections of a fibration $X \times_{\mathrm{Pic}(S)} (\mathbf{Z} \times \mathrm{BPL}) \to X$, having fiber G/PL. We have a map of spaces over X

$$X \times_{\mathrm{Pic}(S)} (\mathbf{Z} \times \mathrm{BPL}) \to X'$$

having homotopy fiber $K(\mathbf{Z}/2\mathbf{Z}, 3)$. Consequently, every section s of the map $X' \to X$ determines a fibration $Y \to X$ with fiber $K(\mathbf{Z}/2\mathbf{Z}, 3)$, where $Y = X \times_{X'} (X \times_{\mathrm{Pic}(S)} (\mathbf{Z} \times \mathrm{BPL}))$. This fibration is classified by a map $X \to K(\mathbf{Z}/2\mathbf{Z}, 4)$, which depends on the choice of section s. We therefore obtain a map

$$\mathbb{L}^s(X, \zeta_X)^\times \times_{\Omega^\infty \widehat{\mathbb{L}}(X, \zeta_X)} \{\widehat{\sigma_X}\} \to K(\mathbf{Z}/2\mathbf{Z}, 4)^X,$$

whose fiber can be identified with the structure space $\mathbb{S}^{tn}(X)$.

Let $\mathbb{S}'(X)$ denote the homotopy fiber product $\mathbb{L}^s(X, \zeta_X)^\times \times_{\Omega^\infty \mathbb{L}^{vs}(X, \zeta_X)} \{\sigma_X^{vs}\}$. Recall that if M is a manifold equipped with a homotopy equivalence $f : M \to X$, then f determines a lifting of σ_X^{vs}, giving a point of $\mathbb{S}'(X)$. Elaborating on this construction, we get a map $\mathbb{S}(X) \to \mathbb{S}'(X)$, which fits into a commutative diagram

$$\begin{array}{ccccc}
\mathbb{S}(X) & \longrightarrow & \mathbb{S}^n(X) & \longrightarrow & \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{S}'(X) & \longrightarrow & \mathbb{L}^s(X, \zeta_X)^\times \times_{\Omega^\infty \widehat{\mathbb{L}}(X, \zeta_X)} \{\widehat{\sigma}_X\} & \longrightarrow & \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X).
\end{array}$$

Assume that the dimension of X is at least 5. The main theorem of this course asserts that the upper row is a fiber sequence, and the bottom row is obviously a fiber sequence. Since the right vertical map is a homotopy equivalence, we see that the square on the left is homotopy Cartesian. Combining this with the above analysis, we obtain:

Theorem 2. *Let X be a Poincare space of dimension ≥ 5. Then we have a fiber sequence of spaces*

$$\mathbb{S}(X) \to \mathbb{S}'(X) \to K(\mathbf{Z}/2\mathbf{Z}, 4)^X,$$

where $\mathbb{S}'(X) = \mathbb{L}^s(X, \zeta_X)^\times \times_{\Omega^\infty \mathbb{L}^{vs}(X,\zeta_X)} \{\sigma_X^{vs}\}$.

Remark 3. It is possible to prove the main results of this course in the setting of topological, rather than piecewise linear manifolds. However, things work slightly differently in low degrees: we actually get a homotopy equivalence from G/Top to the base point component of $L^q(\mathbf{Z})$. The analysis above shows that $\mathbb{S}'(X)$ can be identified with the *topological* structure space of X, parametrizing h-cobordism classes of compact topological manifolds in the homotopy type of X. The map $\psi : \mathbb{S}'(X) \to K(\mathbf{Z}/2\mathbf{Z}, 4)^X$ is a version of the *Kirby-Siebenmann obstruction*: it assigns to every topological manifold M of dimension ≥ 5 a cohomology class $\eta \in \mathrm{H}^4(M; \mathbf{Z}/2\mathbf{Z})$, which vanishes if and only if M admits a PL structure.