The Homotopy Groups of G/PL (Lecture 36)

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Let X be a Poincare space of dimension n. Recall that the normal structure space $\mathbb{S}^n(X)$ is homotopy equivalent to $\mathbb{S}^{tn}(X)$, which is given by the homotopy fiber of the map $\mathrm{BPL}^X \to \mathrm{Pic}(S)^X$ (which classifies stable PL reductions of the Spivak normal bundle of X). The map $\mathrm{BPL} \to \mathrm{Pic}(S)$ is a map of infinite loop spaces, whose fiber we denote by G/PL (here one should think of $G = \mathrm{GL}_1(S)$ as the automorphism group of the sphere spectrum). Consequently, if nonempty, the normal structure space $\mathbb{S}^{tn}(X)$ is homotopy equivalent to a torsor for the infinite loop space $(G/PL)^X$. Our goal in this lecture is to describe the homotopy type of G/PL.

More generally, if $(X, \partial X)$ is a Poincare pair such that ∂X is a compact PL manifold, we can describe $\mathbb{S}^{tn}(X)$ as the homotopy fiber of the canonical map

$$\mathrm{BPL}^X \to \mathrm{Pic}(S)^X \times_{\mathrm{Pic}(S)^{\partial X}} \mathrm{BPL}^{\partial X}$$

(over the point classifying the Spivak bundle of X together with its PL reduction on ∂X). Assume that the PL tangent bundle to ∂X is stably trivial, and that this trivialization extends to a trivialization of the Spivak bundle of X. Then $\mathbb{S}^n(X)$ can be identified with the space of maps of pairs from $(X, \partial X)$ to $(G/PL, *)$. Taking $X = D^n$ and $\partial X = S^{n-1}$, we obtain a canonical homotopy equivalence $\mathbb{S}^{tn}(X) = \Omega^n G/PL$.

Recall that in the above situation, we have a homotopy commutative diagram

$$\begin{array}{ccc} \mathbb{S}(X) & \longrightarrow & \mathbb{S}^{tn}(X) \\ \downarrow & & \downarrow \theta_n \\ * & \longrightarrow & \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X). \end{array}$$

In the special case $(X, \partial X) = (D^n, S^{n-1})$, the Spivak bundle ζ_X is the constant sheaf with value $\Sigma^{-n} S$. Since X is simply connected, it follows that $\mathbb{L}^{vq}(X, \zeta_X)$ is given by $\Sigma^{-n}\mathbb{L}^q(\mathbf{Z})$. We therefore have $\Omega^\infty \mathbb{L}^{vq}(X, \zeta_X) = \Omega^n L^q(\mathbf{Z})$, where $L^q(\mathbf{Z})$ denotes the zeroth space of the spectrum $\mathbb{L}^q(\mathbf{Z})$.

Taking $n = 0$, we obtain a map $\theta : G/PL \to L^q(\mathbf{Z})$. With a little bit of effort, one can show that *all* of the maps θ_n appearing above are induced by θ by passing to n-fold loop spaces. We would like to use the map θ to obtain information about the homotopy type of G/PL. We will obtain this information by combining two facts:

(a) If $n \geq 5$, then we have a homotopy pullback diagram

$$\begin{array}{ccc} \mathbb{S}(D^n) & \longrightarrow & \Omega^n(G/PL) \\ \downarrow & & \downarrow \Omega^n\theta \\ * & \longrightarrow & \Omega^n L^q(\mathbf{Z}). \end{array}$$

This is a special case of our main theorem.

(b) If $n \geq 5$, the structure space $\mathbb{S}(D^n)$ is contractible.

Let us provide an argument for (b). Recall that the structure space $\mathbb{S}(D^n)$ is given as the geometric realization of a simplicial space $\mathbb{S}(D^n)_\bullet$. Let $Y_\bullet$ be the constant simplicial space which consists of a single point in each degree. We claim that the map $\mathbb{S}(D^n)_\bullet \to Y_\bullet$ is a trivial Kan fibration. In other words, we claim that for every integer k, the map $\mathbb{S}(D^n)_k \to \mathbb{S}(D^n)_\bullet(\partial\,\Delta^k)$ is surjective on connected components.

When $k = 0$, this says that $\mathbb{S}(D^n)$ is nonempty: this is obvious, since D^n is already a PL manifold. We now give the proof when $k = 1$; the proof of the general case is the same. Suppose we are given an element of $\mathbb{S}(D^n)_\bullet(\partial\,\Delta^1)$, consisting of two contractible PL manifolds M and M' having boundary S^{n-1}. To lift to a point of $\mathbb{S}(D^n)_1$ we must write an h-cobordism from M to M', trivial along S^{n-1}. This is equivalent showing that the manifold $M \coprod_{S^{n-1}} M'$ bounds a contractible PL manifold of dimension $n+1$. Since M and M' are contractible, $M \coprod_{S^{n-1}} M'$ is homotopy equivalent to a sphere S^n. Since $n \geq 5$, the generalized Poincare conjecture implies that $M \coprod_{S^{n-1}} M'$ is PL homeomorphic to S^n, which bounds the disk D^{n+1}.

Remark 1. In fact, something much stronger is true: each of the spaces $\mathbb{S}(D^n)_k$ is contractible. One can prove this by combining the generalized Poincare conjecture with the Alexander trick.

Warning 2. In proving (b), it is important that we work in the PL category rather than the smooth category. The smooth structure space of D^n is generally not contractible because of the existence of exotic spheres. We can appreciate the importance of the PL condition by examining Smale's proof of the generalized Poincare conjecture. Let M be a manifold of dimension n which is homotopy equivalent to a sphere, and assume that $n \geq 6$ (the case $n = 5$ requires additional effort). Choose two distinct points $x, x' \in M$, and let D and D' be disjoint small disks around x and x', respectively. Let M° be the manifold obtained by removing the interiors of D and D'. The condition that M is homotopy equivalent to S^n guarantees that M° is an h-cobordism from $\partial\,D$ to $\partial\,D'$. The h-cobordism theorem then gives $M^\circ \simeq S^{n-1} \times [0,1]$. If we work in the PL category, we can recover M from M° by "collapsing" the two ends, thereby obtaining $M \simeq S^n$. In the smooth category, this argument does not apply: given a manifold with a boundary sphere, there is no canonical way to assign a smooth structure to the manifold obtained by the collapsing the sphere.

Combining observations (a) and (b), we conclude the following:

(c) If $n \geq 5$, the homotopy fiber of the map $\Omega^n(\theta) : \Omega^n G/PL \to \Omega^n L^q(\mathbf{Z})$ is contractible. In other words, the map

$$\pi_i G/PL \to \pi_i L^q(\mathbf{Z})$$

is injective when $i = 5$, and bijective for $i > 5$.

In fact, we can do a little bit better. Since $\pi_5 L^q(\mathbf{Z})$ is trivial, the fact that $\pi_5 G/PL \to \pi_5 L^q(\mathbf{Z})$ is injective implies that $\pi_5 G/PL$ is trivial. It follows that $\Omega^5(\theta)$ is a homotopy equivalence.

Combining (c) with our calculation of the homotopy groups $L^q_n(\mathbf{Z})$, we obtain the following:

Corollary 3. *Let $n \geq 5$. Then we have a canonical isomorphism*

$$\pi_n G/PL \simeq \begin{cases} 8\mathbf{Z} & \text{if } n = 4k \\ 0 & \text{if } n = 4k+1 \\ \mathbf{Z}/2\mathbf{Z} & \text{if } n = 4k+2 \\ 0 & \text{if } n = 4k+3. \end{cases}$$

Let us now turn our attention to calculating the homotopy groups of G/PL in *low* degrees. We have maps

$$\mathbf{Z} \times \mathrm{BO} \to \mathbf{Z} \times \mathrm{BPL} \to \mathrm{Pic}(S),$$

giving rise to a fiber sequence of spaces

$$PL/O \to G/O \to G/PL.$$

Smoothing theory gives an identification of $\pi_n PL/O$ with the collection of smooth structures on S^n (compatible with the standard PL structure on S^n. It follows that the map

$$\pi_n G/O \to \pi_n G/PL$$

is bijective provided that the PL spheres S^n and S^{n-1} admit unique smoothings. This is true for $n \leq 6$.

We can use this observation to compute $\pi_n G/PL$ for small values of n, since the homotopy groups of $\mathrm{Pic}(S)$ and $\mathbf{Z} \times \mathrm{BO}$ are known. Let us summarize them in the following table:

	$\mathbf{Z} \times \mathrm{BO}$	$\mathrm{Pic}(S)$
π_5	0	0
π_4	$\mathbf{Z}$	$\mathbf{Z}/24\mathbf{Z}$
π_3	0	$\mathbf{Z}/2\mathbf{Z}$
π_2	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$
π_1	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$
π_0	$\mathbf{Z}$	$\mathbf{Z}$.

The map from the groups on the left to the groups on the right is given by the J-homomorphism. This map is an isomorphism on π_1 and π_2 and a surjection on π_4. We therefore obtain

$$\pi_n G/PL = \begin{cases} 24\mathbf{Z} & \text{if } n = 4 \\ 0 & \text{if } n = 3 \\ \mathbf{Z}/2\mathbf{Z} & \text{if } n = 2 \\ 0 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

The map $\pi_n(G/PL) \to \pi_n L^q(\mathbf{Z})$ is obviously an isomorphism if $n = 1$ or $n = 3$, since both sides vanish. Let us show that it is an isomorphism when $n = 2$. Let M be an oriented surface equipped with a spin structure (or "theta characteristic"). Using the spin structure, we can trivialize the tangent bundle of M outside of a small disk $D \subseteq M$, thereby obtaining a degree one normal map $f : M \to D/\partial D \simeq S^2$, which in turn represents an element in $\pi_0 \mathbb{S}^n(S^2) \simeq \pi_0 (G/PL)^{S^2} \simeq \pi_2(G/PL)$. The choice of spin structure determines a quadratic refinement q of the intersection form on $\mathrm{H}^1(M; \mathbf{Z}/2\mathbf{Z})$, making $\mathrm{H}^1(M; \mathbf{Z}/2\mathbf{Z})$ into a nondegenerate quadratic space over the finite field $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$. To prove that the map

$$\mathbf{Z}/2\mathbf{Z} \simeq \pi_2(G/PL) \to \pi_2 L^q(\mathbf{Z}) \simeq \pi_2 L^q(\mathbf{F}_2) \simeq W(\mathbf{F}_2) \simeq \mathbf{Z}/2\mathbf{Z}$$

is nontrivial, it suffices to show that we can choose M (and its spin structure) so that the quadratic space $(\mathrm{H}^1(M; \mathbf{F}_2), q)$ has Arf invariant 1. This is always possible. The collection of spin structures on M is a torsor for $\mathrm{H}^1(M;\mathbf{F}_2)$. If q has Arf invariant 0, then we can modify its Arf invariant by modifying the spin structure by an element $v \in \mathrm{H}^1(M;\mathbf{F}_2)$ satisfying $q(v) = 1$ (which always exists provided that the genus of M is positive).

Let us now compute the map
$$\pi_4 G/PL \to \pi_4 L^q(\mathbf{Z}) = 8\mathbf{Z}$$
determined by θ. Choose a map $u : S^4 \to G/O$, representing a generator of $[u] \in \pi_4 G/O \simeq \pi_4 G/PL$. Then u determines a point of the structure space $\mathbb{S}^{tn}(S^4)$, corresponding to a PL (in fact smooth) 4-manifold M equipped with a degree one normal map $f : M \to S^4$. By construction, the image of $[u]$ in $L^q_4(\mathbf{Z}) \simeq 8\mathbf{Z}$ is the difference of signatures $\sigma_M - \sigma_{S^4}$. Since the signature of S^4 vanishes, this is just the signature of M. The Hirzebruch signature formula shows that this is given by $\frac{p_1(T_M)}{3}[M]$.

Let $v : S^4 \to \mathrm{BO}$ represent a generator $[v] \in \pi_4 \mathrm{BO} \simeq \mathbf{Z}$. Then the composite map $S^4 \xrightarrow{u} G/O \to \mathrm{BO}$ represents $24[v]$. Let us regard p_1 as an element in the integral cohomology ring $\mathrm{H}^4(\mathrm{BO};\mathbf{Z})$. Since f has degree 1, we can identify $\frac{p_1(T_M)}{[}M]$ with $-24v^*(p_1) \in \mathrm{H}^4(S^4;\mathbf{Z}) \simeq \mathbf{Z}$ (the sign comes from our convention that the map u classifies the normal bundle, rather than the tangent bundle). It follows that the image of $[u]$ is given by $-8v^*(p_1) \in 8\mathbf{Z}$.

To compute $v^*(p_1)$, let us identify S^4 with the quaternionic projective space $\mathbf{H}\mathrm{P}^1$, Then we can take $v : \mathbf{H}\mathrm{P}^1 \to \mathrm{BO}$ to classify the real vector bundle $\mathcal{E}$ underlying quaternionic line bundle $\mathcal{O}(1)$. In particular, $\mathcal{E}$ admits the structure of a complex vector bundle , so that
$$p_1(\mathcal{E}) = -c_2(\mathcal{E} \otimes \mathbf{C}) = -c_2(\mathcal{E} \oplus \overline{\mathcal{E}}) = -c_2(\mathcal{E}) - c_1(\mathcal{E})c_1(\overline{\mathcal{E}}) - c_2(\overline{\mathcal{E}}) = -2c_2(\mathcal{E}) = -2e(\mathcal{E}),$$
where e denotes the Euler class of $\mathcal{E}$. Since $\mathcal{E}$ admits a section having exactly one simple zero, we obtain $v^*(p_1) = -2$.

Corollary 4. *The map $\theta : (G/PL) \to L^q(\mathbf{Z})$ induces an isomorphism $\pi_n(G/PL) \to \pi_n L^q(\mathbf{Z})$ for all $n \neq 0, 4$. The map $\pi_4(G/PL) \to \pi_4 L^q(\mathbf{Z})$ is injective, and its image is the subgroup $16\mathbf{Z} \subseteq 8\mathbf{Z} \simeq \pi_4 L^q(\mathbf{Z})$.*

Remark 5. The failure of the map $\pi_4(G/PL) \to \pi_4 L^q(\mathbf{Z})$ to be surjective is a consequence of Rohlin's theorem: a smooth, compact, spin 4-manifold M has signature divisible by 16.

We can regard Corollary **??** as a calculation of the homotopy groups of the homotopy fiber of the map θ. Since these homotopy groups are concentrated in a single degree, the structure of the homotopy fiber can be explicitly determined:

Corollary 6. *We have a homotopy fiber sequence $K(\mathbf{Z}/2\mathbf{Z}, 3) \to G/PL \to L^q(\mathbf{Z})$*

It is possibly to carry out many of the constructions of this course in the setting of topological manifolds (rather than PL manifolds). Stable topological bundles are classified by a space $\mathbf{Z} \times \mathrm{BTOP}$ fitting into a diagram
$$\mathbf{Z} \times \mathrm{BPL} \to \mathbf{Z} \times \mathrm{BTOP} \to \mathrm{Pic}(S).$$
There is an associated fiber sequence $\mathrm{Top}/PL \to G/PL \to G/\mathrm{Top}$. The map $G/PL \to L^q(\mathbf{Z})$ factors through G/Top, and G/Top is homotopy equivalent to the identity component of $L^q(\mathbf{Z})$. Assuming this, Corollary 6 gives a homotopy equivalence $\mathrm{Top}/PL \simeq K(\mathbf{Z}/2\mathbf{Z}, 3)$. We therefore have a fiber sequence of infinite loop spaces
$$\mathbf{Z} \times \mathrm{BPL} \to \mathbf{Z} \times \mathrm{BTOP} \xrightarrow{\phi} K(\mathbf{Z}/2\mathbf{Z}, 4).$$
The map ϕ classifies the *Kirby-Siebenmann obstruction.* If M is a topological manifold, its stable (topological) tangent bundle is classified by a map $M \to \mathbf{Z} \times \mathrm{BTOP}$. Composing this map with ϕ, we get a map $M \to K(\mathbf{Z}/2\mathbf{Z}, 4)$, classified by a cohomology class $\nu \in \mathrm{H}^4(M;\mathbf{Z}/2\mathbf{Z})$. This class vanishes if and only if the stable tangent bundle of M admits a PL reduction. In particular, ν vanishes whenever M admits a PL structure. One can show that the converse holds if the dimension of M is different from 4.