Surgery in the Middle Dimension (Lecture 35)

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Recall that our goal is to prove the following:

Theorem 1. *Let X be a Poincare space of dimension $n \geq 5$, let ζ be a stable PL bundle on X, and let $f : M \to X$ be a degree one normal map, where M is a compact PL manifold. Assume that M and X are connected and that f induces an isomorphism $\pi_1 M \simeq \pi_1 X \simeq G$, so that σ_f^{vq} can be represented by a Poincare object (V, q), where $V \in \mathrm{LMod}_{\mathbf{Z}[G]}$ satisfies $C_*(\widetilde{M}; \mathbf{Z}) \simeq C_*(\widetilde{X}; \mathbf{Z}) \oplus \Sigma^n V$ (here $\widetilde{M}$ and $\widetilde{X}$ denote universal covers of M and X, respectively).*

Assume that f is p-connected, that we are given a map $u : \Sigma^{p-n}\mathbf{Z}[G] \to V$ a nullhomotopy of $q|\Sigma^{p-n}\mathbf{Z}[G]$, so that (algebraic) surgery along u determines a bordism bordism from (V, q) to another Poincare object (V', q'). Then this (algebraic) bordism can be obtained by performing (geometric) surgery with respect to a normal surgery datum $\alpha : S^p \times D^{q+1} \hookrightarrow M$.

In the last lecture, we explained why this was true in the case where $p < \frac{n}{2}$. Our goal in this lecture is sketch the proof in the case $p = \frac{n}{2}$. Let us therefore assume that $n = 2k$, that f is k-connected, and that we are given a class $u : \Sigma^{-k}\mathbf{Z}[G] \to V$ and a nullhomotopy of $q|\Sigma^{-k}\mathbf{Z}[G]$. We would like to lift this data to a normal surgery datum in M.

Recall that, in order to identify $\mathbb{L}^{vq}(X, \zeta_X)$ with $\Sigma^{-n}\mathbb{L}^q(\mathbf{Z}[G])$, we chose a base point of X and a trivialization of the Spivak bundle of X at the base point. Without loss of generality, we may suppose that this base point is the image a point $x_0 \in M$, that G is given canonically as $\pi_1(M, x_0)$, and that x_0 is the base point used in constructing the universal cover $\widetilde{M}$ (so that x_0 lifts canonically to a point $\widetilde{x}_0 \in \widetilde{M}$). The trivialization of ζ_X at $f(x_0)$ gives an orientation of the manifold M at the point x_0, hence an orientation of $\widetilde{M}$ at the point $\widetilde{x_0}$ (which extends to an orientation of the whole of $\widetilde{M}$, since $\widetilde{M}$ is simply connected). We have a group homomorphism $\epsilon : G \to \{\pm 1\}$, where $\epsilon(g) = 1$ if the action of G on $\widetilde{M}$ preserves the orientation, and $\epsilon(g) = -1$ otherwise.

Let us now try to lift the class u to a normal surgery datum in X. The first step in the argument proceeds just as in the previous lecture. Since f is k-connected, we can apply the relative Hurewicz theorem to the pair (X, M) to represent u by a diagram

$$\begin{array}{ccc} S^k & \longrightarrow & \widetilde{M} \\ \downarrow{\scriptstyle \widetilde{\alpha}_0} & & \downarrow \\ D^{k+1} & \longrightarrow & \widetilde{X}. \end{array}$$

Let $\alpha_0 : S^k \to M$ be the composition of $\widetilde{\alpha}_0$ with the covering map $\widetilde{M} \to M$.

We now encounter our first difficulty: since $n = 2k$, we cannot use general position arguments to ensure that the map $\alpha_0 : S^k \to M$ is an embedding. However, we get *almost* this much for free: we can assume that α_0 is an immersion with a finite number of points of simple self-intersection. Then α_0 is factors as a composition

$$S^k \xrightarrow{\beta} U \to M,$$

where the map $U \to M$ is a local homeomorphism and β is both an embedding and a homotopy equivalence. Since $f \circ \alpha_0$ is canonically nullhomotopic, the pullback $\beta^* - T_M$ is trivialized, so that $-T_M$ is trivial along U. In particular, we can assume that U is equipped with a smooth structure (and a framing). Modifying β by a homotopy if necessary, we may assume that β is a smooth embedding with normal bundle $\mathcal{E}$, classified by a map $S^k \to \mathrm{BO}(k)$. Using the framings of S^k and U, we obtain a nullhomotopy of the composite map $S^k \to \mathrm{BO}(k) \to \mathrm{BO}$. Arguing as in the previous lecture, we can lift this to a nullhomotopy of the composite map $S^k \to \mathrm{BO}(k) \to \mathrm{BO}(k+1)$. Such a nullhomotopy cannot always be lifted to a trivialization of $\mathcal{E}$: we encounter an obstruction given by a map $S^k \to O(k+1)/O(k) \simeq S^k$, which has some degree $d \in \mathbf{Z}$. When k is even, this integer is half of the Euler class of $\mathcal{E}$; when k is odd, the integer d is really only well-defined modulo 2 (that is, the exact integer d depends on a choice of trivialization of $\mathcal{E} \oplus \underline{R}$). In either case, we can get the obstruction to vanish by introducing some "kinks" in the map β (that is, by replacing β by a map which is homotopic but not isotopic to β, possibly introducing some additional double points). By means of this procedure, we can ensure that the stable framing of $\mathcal{E}$ lifts to a trivialization of $\mathcal{E}$.

If the map α_0 were an embedding, then (possibly after shrinking U) we can identify U with an open subset of M, and the above argument allows us to extend α_0 to a normal surgery datum $\alpha : S^k \times D^k \hookrightarrow M$. However, α_0 is generally not an embedding: we certainly cannot arrange this for an arbitrary map $u : \Sigma^{-k} \to V$. We must take advantage of the algebraic information encoded in the nullhomotopy of $q|\Sigma^{-k}\mathbf{Z}[G]$. Note that $q|\Sigma^{-k}\mathbf{Z}[G]$ can be regarded as a point of the space $\Omega^{\infty+2k}Q^q(\mathbf{Z}[G]) = \Omega^{\infty}(\mathbf{Z}[G])_{h\Sigma_2}$. Here the action of the permuation group Σ_2 on $\mathbf{Z}[G]$ is given by $g \mapsto (-1)^k \epsilon(g) g^{-1}$, and the group of connected components of $\Omega^{\infty}(\mathbf{Z}[G])$ is given by the abelian group of coinvariants $\mathbf{Z}[G]_{\Sigma_2}$. The homotopy class of $q|\Sigma^{-k}\mathbf{Z}[G]$ is an element of this abelian group depending on u; let us denote it by $q(u)$.

Let us describe the image of $q(u)$ under the norm map

$$\mathrm{tr} : \mathbf{Z}[G]_{\Sigma_2} \to \mathbf{Z}[G]^{\Sigma_2} \subseteq \mathbf{Z}[G].$$

Unwinding the definition, we see that this is given by restricting the $\mathbf{Z}[G]$-valued intersection form on $C_*(\widetilde{M}; \mathbf{Z})$ along u. In particular, we have $\mathrm{tr}(q(u)) = \sum_{g \in G} j(g) g$, where $j(g)$ is the number of points of intersection (counted with multiplicity) of $\widetilde{\alpha}_0(S^k)$ with its translate under g. Let us describe this number more explicitly. Let $Y \subseteq S^k$ be the (finite) set of points over which $\alpha_0 : S^k \to M$ fails to be an immersion. For each $y \in Y$, there exactly one other element $\widehat{y} \in Y$ such that $\alpha_0(y) = \alpha_0(\widehat{y})$. We then have $\widetilde{\alpha}_0(\widehat{y}) = g_y \widetilde{\alpha}_0(y)$ for some element $g_y \in G$. Moreover, the orientations of S^k at y and $\widehat{y}$ determine an orientation of $\widetilde{M}$ at $\widetilde{\alpha}_0(\widehat{y})$. Define $\eta(y) \in \{\pm 1\}$ so that $\eta(y) = 1$ if this orientation agrees with our given orientation on $\widetilde{M}$, and $\eta(y) = -1$ otherwise. Note that $g(\widehat{y}) = g(y)^{-1}$, and that $\eta(\widehat{y}) = (-1)^k \epsilon(g(y))$ (the first sign comes from the fact that permuting the factors of $\mathbb{R}^k \times \mathbb{R}^k$ is orientation-reversing when k is odd, and the second from the fact that translation by $g(y)$ is orientation-reversing when $\epsilon(g(y)) = -1$). Unwinding the definitions (and using the triviality of the normal bundle $\mathcal{E}$), we see that the self-intersection of the homology class represented by $\widetilde{\alpha}_0$ is given by the expression

$$E = \sum_{y \in Y} \eta(y) g(y).$$

Note that $\sum_{y \in Y} \eta(y) g(y) = \sum_{y \in Y} \eta(\widehat{y})(-1)^k \epsilon(g(y)) g(\widehat{y})^{-1}$. In particular, this sum is invariant under the involution $g \mapsto (-1)^k \epsilon(g) g^{-1}$, on $\mathbf{Z}[G]$, as we know it must be.

We now observe that E lies in the image of the transfer map $\mathrm{tr} : \mathbf{Z}[G]_{\Sigma_2} \to \mathbf{Z}[G]^{\Sigma_2}$. Namely, choose a decomposition $Y = Y_- \coprod Y_+$, where for every element $y \in Y$ exactly one of the points $\{y, \widehat{y}\}$ belongs to Y_-. Then write $E_0 = \sum_{y \in Y_-} \eta(y) g(y)$. Then $E_0 \in \mathbf{Z}[G]$ determines an element of $\mathbf{Z}[G]_{\Sigma_2}$, whose transfer is given by E. The specific element $E_0 \in \mathbf{Z}[G]$ depends on the choice of decomposition $Y = Y_- \cup Y_+$, but its image in $\mathbf{Z}[G]_{\Sigma_2}$ does not.

We assert the following without proof:

Claim 2. The element E_0 defined above is a representative for $q(u) \in \mathbf{Z}[G]_{\Sigma_2}$.

Remark 3. When X is simply connected and n is divisible by 4, the transfer map $\mathrm{tr} : \mathbf{Z}[G]_{\Sigma_2} \simeq \mathbf{Z} \to \mathbf{Z} \simeq \mathbf{Z}[G]^{\Sigma_2}$ is injective (it is given by multiplication by 2), so that Claim 2 follows from our analysis of the intersection pairing).

Let us now study the meaning of the condition that $q(u) = 0$. Let $G_{(2)} \subseteq G$ be the subset consisting of 2-torsion elements. Write $G_{(2)} = G^{-}_{(2)} \cup G^{+}_{(2)}$, where $g \in G^{+}_{(2)}$ if $(-1)^k \epsilon(g) = 1$, and $g \in G^{-}_{(2)}$ if $(-1)^k \epsilon(g) = -1$. Choose a subset $H \subseteq G - G_{(2)}$ such that for every element $g \in G$ which is no 2-torsion, exactly one of the elements $\{g, g^{-1}\}$ belongs to H. Then $\mathbf{Z}[G]_{\Sigma_2}$ is given by the direct sum

$$(\bigoplus_{g \in H} \mathbf{Z}g) \oplus (\bigoplus_{g \in G^{+}_{(2)}} \mathbf{Z}g) \oplus (\bigoplus_{g \in G^{-}_{(2)}} \mathbf{Z}/2\mathbf{Z}g).$$

We may assume that Y_- is chosen such that for $y \in Y_+$, if $g(y)$ is not 2-torsion, then $g(y) \in H$. Then $q(u) = 0$ implies the following:

(i) If $g \in H$, then the sum $\sum_{y \in Y_-, g(y) = g} \eta(y)$ vanishes.

(ii) If $g \in G^{+}_{(2)}$, then the sum $\sum_{y \in Y_-, g(y) = g} \eta(y)$ vanishes.

(iii) If $g \in G^{-}_{(2)}$, then the sum $\sum_{y \in Y_-, g(y) = g} \eta(y)$ is even.

Note that when $g(y) \in G^{-}_{(2)}$, then replacing y by $\widehat{y}$ changes the sign $\eta(y)$. We therefore see that $q(u) = 0$ if and only if we can choose Y_- such that the sum E_0 vanishes. In this case, we can partition Y into subsets of the form $\{y, \widehat{y}, y', \widehat{y}'\}$ where $g(y) = g(y')$ and $\eta(y) = -\eta(y')$.

The argument now proceeds by invoking Whitney's trick for cancelling double points. Suppose we are given a quadruple $\{y, \widehat{y}, y', \widehat{y}'\}$ as above. Choose a path γ from y to y' in S^p, and another path $\widehat{\gamma}$ from $\widehat{y}'$ to $\widehat{y}$. We may assume that the paths γ and $\widehat{\gamma}$ are embedded, do not intersect one another, and do not meet any points of Y except at their endpoints.

After applying the map α_0, we obtain paths in M which can be concatenated to yield a closed loop. The condition that $g(y) = g(y')$ implies that this loop is nullhomotopic (it lifts a loop in $\widetilde{M}$, given by concatenating $\widetilde{\alpha}_0(\gamma)$ with the translate by $g(y)^{-1}$ of $\widetilde{\alpha}_0(\widehat{\gamma})$). We may therefore represent this loop by the restriction to the boundary of a continuous map $D^2 \to M$. Since the dimension of M is at least 5 (in fact, at least 6, since $n = 2k$ is even), we may assume that the map $D^2 \to M$ is an embedding which does not intersect the image of α_0 except at the boundary. By analyzing a small neighborhood of the image of D^2, we can "push" the immersion α_0 to obtain a new (isotopic) immersion, in which the double points $\{y, y', \widehat{y}, \widehat{y}'\}$ have been eliminated. Iterating this procedure, we can reduce to the case where $Y = \emptyset$, so that α_0 is an embedding as desired.

The above analysis allows us to construct a normal surgery datum $\alpha : S^k \times D^k \hookrightarrow M$ representing any map $u : \Sigma^{-k}\mathbf{Z}[G] \to V$ such that $q(u) = 0$. However, this is not quite enough to prove Theorem **??**. Let T denote the collection of homotopy classes of trivializations of $q|\Sigma^{-k}\mathbf{Z}[G]$. Then T is a torsor for the fundamental group $\pi_1 \Omega^{\infty+n} Q^q(\Sigma^{-k}\mathbf{Z}[G]) \simeq \mathrm{H}_1(\Sigma_2; \mathbf{Z}[G])$, where the permutation group Σ_2 acts on $\mathbf{Z}[G]$ as indicated above. A simple calculation shows that this group is given by the direct sum $A = \bigoplus_{g \in G^{+}_{(2)}} \mathbf{Z}/2\mathbf{Z}$. Every normal surgery datum α representing u determines a nullhomotopy of $q|\Sigma^{-k}\mathbf{Z}[G]$, giving an element $t(\alpha) \in T$. In order to prove Theorem **??**, we need to show that every element of T has the form $t(\alpha)$, for some normal surgery datum α. For this, we need a procedure to modify a given normal surgery datum α to change the invariant $t(\alpha)$ in a controlled way.

The above analysis suggests such a procedure. Let $g \in G^{+}_{(2)}$. Suppose we begin with a fixed embedding $\alpha_0 : S^k \to M$. By "reversing" the Whitney trick described above, we can modify α_0 by an isotopy to obtain an immersion $\alpha'_0 : S^k \to M$ which fails to be an embedding at a set of points $\{y, y', \widehat{y}, \widehat{y}'\}$, satisfying $g(y) = g(y') = g$ and $\eta(y) = -\eta(y')$. Since $g = g^{-1}$ and $(-1)^k \epsilon(g) = 1$, we have $g(y) = g(\widehat{y})$ and $\eta(y) = \eta(\widehat{y})$. We may therefore switch the roles of y and $\widehat{y}$ in Whitney's construction, to obtain a *different* isotopy from

α_0' to an embedding α_0''. Since α_0 and α_0'' are isotopic (though immersions), every normal surgery datum α extending α_0 determines a normal surgery datum α'' extending α_0''. We assert that $t(\alpha) = t(\alpha'') + x$, where x is a generator for the summand of A corresponding to the element $g \in G$. Since the collection of such elements generate the abelian group A and A acts transitively on T, this claim completes the proof of Theorem 1.