(Geometric) Surgery Below the Middle Dimension (Lecture 34)

April 20, 2011

In the last lecture, we reduced the main theorem of this course to the following assertion:

Theorem 1. *Let X be a Poincare space of dimension $n \geq 5$, let ζ be a stable PL bundle on X, and let $f : M \to X$ be a degree one normal map, where M is a compact PL manifold. Assume that M and X are connected and that f induces an isomorphism $\pi_1 M \simeq \pi_1 X \simeq G$, so that σ_f^{vq} can be represented by a Poincare object (V, q), where $V \in \mathrm{LMod}_{\mathbf{Z}[G]}$ satisfies $C_*(\widetilde{M}; \mathbf{Z}) \simeq C_*(\widetilde{X}; \mathbf{Z}) \oplus \Sigma^n V$ (here $\widetilde{M}$ and $\widetilde{X}$ denote universal covers of M and X, respectively).*

Assume that f is p-connected, that we are given a map $u : \Sigma^{p-n}\mathbf{Z}[G] \to V$ a nullhomotopy of $q|\Sigma^{p-n}\mathbf{Z}[G]$, so that (algebraic) surgery along u determines a bordism bordism from (V, q) to another Poincare object (V', q'). Then this (algebraic) bordism can be obtained by performing (geometric) surgery with respect to a normal surgery datum $\alpha : S^p \times D^{q+1} \hookrightarrow M$.

In this lecture, we will treat the "easy" case of Theorem 1, where p is strictly smaller than $\frac{n}{2}$. Let $f : M \to X$ be as in Theorem 1. Assume that f is p-connected, and choose any map $e : \Sigma^p \mathbf{Z}[G] \to \Sigma^n V$, classified by an element of $\pi_{p-n} V \simeq \ker(\mathrm{H}_p(\widetilde{M}; \mathbf{Z}) \to \mathrm{H}_p(\widetilde{X}; \mathbf{Z})) = \mathrm{H}_{p+1}(\widetilde{X}, \widetilde{M}; \mathbf{Z})$. Since f is p-connected, the Hurewicz theorem allows us to identify this group with the relative homotopy group $\pi_{p+1}(\widetilde{X}, \widetilde{M})$: that is, with $\pi_p F$, where F denotes the homotopy fiber of the map $\widetilde{f} : \widetilde{M} \to \widetilde{X}$ (note that F can also be identified with the homotopy fiber of the map $f : M \to X$. Consequently, e determines a map $\alpha_0 : S^p \to M$, together with a nullhomotopy of the composite map $S^p \to M \to X$.

Since $p < \frac{n}{2}$, we can assume (modifying the map $\overline{e}$ by a homotopy if necessary) that α_0 is an embedding. Note that we have an equivalence of stable PL bundles

$$\alpha_0^*(-T_M) \simeq \overline{e}^* f^* \zeta.$$

Since the composition $f \circ \alpha_0$ is nullhomotopic, we can trivialize the (stable) tangent bundle of M in a neighborhood of the image of α_0. It follows from smoothing theory that M admits a smooth structure in a neighborhood U of $\alpha_0(S^p)$, and this smooth structure admits a framing. Modifying α_0 by a homotopy if necessary, we may assume that it is a smooth embedding from S^p into U. This embedding has a normal bundle, which we will denote by $\mathcal{E}$. Since U and S^p are framed, the bundle $\mathcal{E}$ is framed: that is, it is stably trivial. In order to extend α_0 to a normal surgery datum $\alpha : S^p \times D^{n-p} \hookrightarrow M$, we need to promote this stable trivialization of $\mathcal{E}$ to an actual trivialization of $\mathcal{E}$.

The bundle $\mathcal{E}$ is classified by a map $\chi : S^p \to \mathrm{BO}(n-p)$. Our stable framing gives a nullhomotopy of the composite map $S^p \xrightarrow{\chi} \mathrm{BO}(n-p) \to \mathrm{BO}(N)$ for N large. We wish to lift this to a nullhomotopy of the map χ itself. We can carry out this lifting in stages. Suppose that we have a map $\psi : S^p \to \mathrm{BO}(k)$ and a trivialization of the composite map $S^p \to \mathrm{BO}(k) \to \mathrm{BO}(k+1)$. This nullhomotopy determines a factorization of ψ through the homotopy fiber of the map $\mathrm{BO}(k) \to \mathrm{BO}(k+1)$, which is homotopy equivalent to the quotient $O(k+1)/O(k) \simeq S^k$. If $p < k$, such a map is automatically nullhomotopic, and therefore the lifting is possible. This analysis applies in our case of interest: for any $k \geq n - p$, we have $p \leq k$, since we have assumed that $2p < n$.

The above argument shows that every map $e : \Sigma^p \mathbf{Z}[G] \to \Sigma^n V$ can be lifted to a map $\alpha_0 : S^p \to M$ which extends to a normal surgery datum $\alpha : S^p \times D^{n-p} \hookrightarrow M$. However, this is not quite sufficient to prove Theorem 1. A normal surgery datum determines both a map $e : \Sigma^p \mathbf{Z}[G] \to \Sigma^n V$ *and* a nullhomotopy of $q|\Sigma^{p-n}\mathbf{Z}[G]$. In the situation of Theorem 1, there is generally no guarantee that this coincides with the nullhomotopy we are interested in.

Note that $q|\Sigma^{p-n}\mathbf{Z}[G]$ can be identified with a point in the 0th space of the spectrum

$$\Sigma^{-n} Q^q(\Sigma^{p-n}\mathbf{Z}[G]) = \Sigma^{-n}(\Sigma^{2n-2p}\mathbf{Z}[G])_{h\Sigma_2} = \Sigma^{n-2p}(\mathbf{Z}[G])_{h\Sigma_2},$$

where on the right hand side the permutation group acts on $\mathbf{Z}[G]$ by means of the involution of the previous lecture together with the sign $(-1)^{n-p}$. Since $n > 2p$, this spectrum is always connected. If $n > 2p+1$, this spectrum is simply connected: it follows that a nullhomotopy of $q|\Sigma^{p-n}\mathbf{Z}[G]$ is uniquely determined, up to homotopy. This completes the proof of Theorem 1 in this case.

If $n = 2p+1$, we need to work a little bit harder. Let T be the set of homotopy classes of trivializations of $q|\Sigma^{p-n}\mathbf{Z}[G]$. Then T is a torsor for the group $\pi_1 \Sigma^{-n} Q^q(\Sigma^{p-n}\mathbf{Z}[G]) \simeq \mathbf{Z}[G]_{\Sigma_2}$, where Σ_2 acts on the group $\mathbf{Z}[G]$ as indicated above. In particular, there is a transitive action of the group ring $\mathbf{Z}[G]$ (regarded as an abelian group under addition) on the set T. If $\alpha : S^p \times D^{n-p} \hookrightarrow M$ is a normal surgery datum whose restriction α_0 to $S^p \times \{0\}$ represents the homology class $e \in \pi_{p-n} V$, then α determines a nullhomotopy of $q|\Sigma^{p-n}\mathbf{Z}[G]$. Let us denote the corresponding element of T by $t(\alpha)$. To prove Theorem 1, it will suffice to show that for any element $x \in \mathbf{Z}[G]$, we can find another normal surgery datum α' (still representing the homology class e) such that $t(\alpha') = x + t(\alpha)$. Since $\mathbf{Z}[G]$ is generated as abelian group by the elements of G, we may assume that $x = \pm g$, for some element $g \in G$. In this case, we assert without proof that there is a specific geometric construction for obtaining the surgery datum α' (it is determined by writing an isotopy through immersions from α_0 to another embedding α_0'). (Some pictures are provided in class; we will not reproduce them here.)