From Algebraic to Geometry Surgery (Lecture 33)

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Recall that our goal is to prove the following:

Theorem 1. *Let X be a Poincare pair of dimension $n \geq 5$, ζ a stable PL bundle on X, and $f : M \rightarrow X$ a degree one normal map, where M is a PL manifold. Let $\sigma_f^{vq} \in \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$ be the relative signature of f, and suppose we are given a path p from σ_f^{vq} to the base point of $\Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$. (We can identify such a path with a Lagrangian in the Poincare object representing σ_f^{vq}, which is well-defined up to bordism). Then there exists a Δ^1-family of degree one normal maps $F : B \rightarrow X \times \Delta^1$, where B is a bordism from $M = F^{-1}(X \times \{0\})$ to a PL manifold $N = F^{-1}(X \times \{1\})$ such that F induces a homotopy equivalence $f' : N \rightarrow X$. Moreover, we can arrange that F determines a path from σ_f^{vq} to $\sigma_{f'}^{vq} = 0$ which is homotopic to p.*

In the last lecture, we introduced the technique of *surgery* as a method of producing normal bordisms from M to other PL manifolds (equipped with degree one normal maps to X). Moreover, we saw how to use the method of surgery to reduce Theorem 1 to the special case where $f : M \rightarrow X$ induces an equivalence of fundamental groupoids. We may further assume without loss of generality that X (and therefore also M) are connected. Let us fix a base point of X, allowing us to define a fundamental group $G = \pi_1 X$. Let $\widetilde{X}$ denote the universal cover of X and let $\widetilde{M} = M \times_X \widetilde{X}$ be the corresponding universal cover of M, so that G acts on $\widetilde{X}$ and $\widetilde{M}$ by deck transformations.

The spherical fibration ζ_X is classified by a map $X \rightarrow \mathrm{Pic}(S)$, which induces a map

$$G = \pi_1 X \rightarrow \pi_1 \mathrm{Pic}(S) = \pi_0 \mathrm{GL}_1(S) = \mathrm{GL}_1(\pi_0 S) = \mathrm{GL}_1(\mathbf{Z}) = \{\pm 1\},$$

which we will denote by ϵ. This homomorphism vanishes if and only if ζ_X is orientable with respect to ordinary homology (that is, if and only if $\zeta_X \wedge \mathbf{Z}$ is a constant sheaf). Let $\mathbf{Z}[\pi_1 X] = \mathbf{Z}[G]$ be the group algebra of G. Then $\mathbf{Z}[G]$ admits an involution, given by $g \mapsto \epsilon(g) g^{-1}$. Let $Q^s, Q^q : (\mathrm{LMod}^{fp}_{\mathbf{Z}[G]})^{op} \rightarrow \mathrm{Sp}$ be the quadratic functors given by

$$Q^q(M) = \mathrm{Mor}_{\mathbf{Z}[G]-\mathbf{Z}[G]}(M \wedge M, \mathbf{Z}[G])_{h\Sigma_2} \qquad Q^s(M) = \mathrm{Mor}_{\mathbf{Z}[G]-\mathbf{Z}[G]}(M \wedge M, \mathbf{Z}[G])^{h\Sigma_2}.$$

Using the π-π theorem, we can identify $\Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$ with $\Omega^{\infty+n} \mathbb{L}^q(\mathbf{Z}[G]) \simeq L(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}[G]}, \Omega^n Q^q)$.

Let us attempt to describe the invariant σ_f^{vq} more explicitly in these terms. The visible symmetric signatures σ_X^{vs} and σ_M^{vs} determine Poincare objects of $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}[G]}, \Omega^n Q^s)$. Unwinding the definitions, we see that these objects are given concretely by the *duals* of the $\mathbf{Z}[G]$-modules given by $C_*(\widetilde{X}; \mathbf{Z})$ and $C_*(\widetilde{M}; \mathbf{Z})$ (note that each of these is a finitely presented $\mathbf{Z}[G]$-module, since $\widetilde{X}$ and $\widetilde{M}$ admit cell decompositions which are invariant under G, whose cells break up into finitely many free G-orbits). Using Poincare duality, we see that both of these objects are self-dual up to a shift; more precisely, the relevant Poincare objects are represented by $\Sigma^{-n} C_*(\widetilde{X}; \mathbf{Z})$ and $\Sigma^{-n} C_*(\widetilde{M}; \mathbf{Z})$.

Remark 2. Using Poincare duality on the noncompact manifold $\widetilde{M}$, we can identify $\Sigma^{-n} C_*(\widetilde{M}; \mathbf{Z})$ with $C^*_c(\widetilde{M}; \mathbf{Z})$, where the subscript indicates that we take compactly supported cochains. This identification is not quite G-equivariant, since the action of G on $\widetilde{M}$ may not preserve orientations (the failure of the G-action to preserve orientations is codified by the homomorphism $\epsilon : G \to \{\pm 1\}$). Informally speaking, the symmetric bilinear form on $C^*_c(\widetilde{M}; \mathbf{Z})$ is easy to describe: it carries a pair of compactly supported $\mathbf{Z}$-valued cochains u and v to the sum

$$\sum_{g \in G} (u \cup g(v))[M] \in \mathbf{Z}[G].$$

Here the condition that u and v both have compact support guarantees that the sum on the left hand side is indeed finite.

We have seen that the degree one map $f : M \to X$ determines a homotopy equivalence

$$\Sigma^{-n} C_*(\widetilde{M}; \mathbf{Z}) \simeq V \oplus \Sigma^{-n} C_*(\widetilde{X}; \mathbf{Z})$$

for some (finitely presented) $\mathbf{Z}[G]$-module spectrum V. Moreover, there is a point $q \in \Sigma^{-n} Q^q(V)$ such that (V, q) is a Poincare object of $\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}[G]}$, representing the relative signature σ^{vq}_f.

Now suppose we are given a normal surgery datum in M, giving in particular a codimension zero embedding $\alpha : S^p \times D^{q+1} \hookrightarrow M$. This determines a normal bordism from M to another PL manifold N equipped with a degree one normal map $f' : N \to X$, hence a bordism between the Poincare objects representing σ^{vq}_f and $\sigma^{vq}_{f'}$. The latter bordism is given by an *algebraic* surgery along some map of $\mathbf{Z}[G]$-module spectra $u : \Sigma^{-n} K \to V$. Let $B(\alpha)$ denote the trace of the surgery along α and let $\widetilde{B(\alpha)} = B(\alpha) \times_X \widetilde{X}$. Then $\mathrm{cofib}(u)$ is the $\mathbf{Z}[G]$-module underlying relative signature associated to $B(\alpha)$ (as a normal bordism); that is, we have

$$\Sigma^{-n} C_*(\widetilde{X}; \mathbf{Z}) \oplus \mathrm{cofib}(u) \simeq \Sigma^{-n} C_*(\widetilde{B(\alpha)}; \mathbf{Z}).$$

It follows that $\Sigma^{-n} K \simeq \mathrm{fib}(V \to \mathrm{cofib}(u)) \simeq \mathrm{fib}(\Sigma^{-n} C_*(\widetilde{M}; Z) \to \Sigma^{-n} C_*(\widetilde{B(\alpha)}; \mathbf{Z})$. We have a homotopy pushout diagram of spaces

$$\begin{array}{ccc} S^p & \longrightarrow & D^{p+1} \\ \downarrow & & \downarrow \\ M & \longrightarrow & B(\alpha) \end{array}$$

which lifts to a homotopy pushout diagram of G-spaces

$$\begin{array}{ccc} S^p \times G & \longrightarrow & D^{p+1} \times G \\ \downarrow & & \downarrow \\ \widetilde{M} & \longrightarrow & \widetilde{B(\alpha)}. \end{array}$$

It follows that K is equivalent to the homotopy fiber of the map of the map $C_*(S^p \times G; \mathbf{Z}) \to C_*(D^{p+1} \times G; \mathbf{Z})$, which is homotopy equivalent to $\Sigma^p \mathbf{Z}[G]$. The map $u : \Sigma^{-n} K \to V$ is classified up to homotopy by an element of $\pi_{p-n} V$, which we can regard as a direct summand of $\pi_p C_*(\widetilde{M}; \mathbf{Z}) \simeq \mathrm{H}_p(\widetilde{M}; \mathbf{Z})$. The above calculation shows that this homology class if the Hurewicz image of the class in $\pi_p \widetilde{M}$ determined by a choice of lift of the map $\alpha_0 : S^p \to M$ determined by the surgery datum α.

Remark 3. In the above discussion, the module $K \simeq \Sigma^p \mathbf{Z}[G]$ is determined by the choice of dimension p, and the map $u : \Sigma^{-n} K \to V$ is determined by the homotopy class of the map $\alpha_0 : S^p \to M$ (and a nullhomotopy h of the composite map $S^p \to M \to X$). To perform algebraic surgery on the Poincare object (V, q), we need more: namely, a nullhomotopy of the restriction $q | \Sigma^{-n} K$. This choice of nullhomotopy depends on additional geometric data: the fact that α_0 is an embedding, and a choice of trivial normal bundle to α_0 compatible with h.

The key step in the proof of Theorem 1 is the following, which asserts that there is a sufficient supply of normal surgery data:

Theorem 4. *Let $f : M \to X$ be as in Theorem 1. Assume that M and X are connected and that f induces an isomorphism $\pi_1 M \simeq \pi_1 X \simeq G$, and let (V, q) be defined as above. Assume that f is p-connected, that we are given a map $u : \Sigma^{p-n}\mathbf{Z}[G] \to V$ a nullhomotopy of $q|\Sigma^{p-n}\mathbf{Z}[G]$, so that (algebraic) surgery along u determines a bordism bordism from (V, q) to another Poincare object (V', q'). Then this (algebraic) bordism can be obtained by performing (geometric) surgery with respect to a normal surgery datum $\alpha : S^p \times D^{q+1} \hookrightarrow M$.*

Remark 5. In the situation of Theorem 4, the relevant surgery does not change the fundamental group of M. the relevant p-surgeries do not change the fundamental group of M. Suppose we are given an embedding $\alpha : S^p \times D^{q+1} \hookrightarrow M$ (where $p + q + 1 = n$). The manifold M° obtained from M by removing the interior of the image of α is homotopy equivalent to $M - S^p$, which differs from M in codimension $q + 1 = n - p$. General position arguments show that this procedure does not change the fundamental group of M provided that $n - p \geq 3$. This condition is clearly satisfied when $n \geq 5$ and $p \leq \frac{n}{2}$. Surgery along α produces a new manifold M', which is obtained as a pushout

$$M^{\circ} \coprod_{S^p \times S^q} D^{p+1} \times S^q.$$

Since $q = n - p - 1 \geq 2$, the sphere S^q is simply connected. It follows from van Kampen's theorem $\pi_1 M^{\circ} \to \pi_1 M'$ is surjective. Since the composite map $\pi_1 M^{\circ} \to \pi_1 M' \to \pi_1 X$ is injective, we deduce that $\pi_1 M' \simeq \pi_1 M^{\circ}$.

Our goal for the remainder of this lecture is to explain how to deduce Theorem 1 from Theorem 4. To this end, let us suppose that we are given an arbitrary Lagrangian in (V, q), given by a map $L \to V$ and a nullhomotopy of $q|L$. We would like to show that the Lagrangian L can be obtained by a sequence of normal surgeries on the PL manifold M. Before we can make this assertion, we may need to modify the choice of Lagrangian L. Recall that the data of V together with the Lagrangian L can be identified with a quadratic object (W, q') of $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}[G]}, \Sigma^{-n-1}Q^q)$, where $\Sigma^{-n-1}\mathbb{D}(W) \simeq L$ and q induces a map $W \to \Sigma^{-n-1}\mathbb{D}(W) \simeq L$ having cofiber V. Before proving Theorem 1, we are free to replace L by a cobordant Lagrangian by doing surgery on the quadratic object W. We may therefore assume that W has been simplified by means of (algebraic) surgery below the middle dimension. Write $n = 2k$ or $n = 2k + 1$. We may assume that W $(-k-1)$-connective, so that $\Sigma^n L \simeq \Sigma^{-1}\mathbb{D}(W)$ has projective amplitude $\leq k$. Note in particular that $\Sigma^n W$ is connected, and $\Sigma^n V$ is connected (since $\mathrm{H}_0(\widetilde{M}; \mathbf{Z}) \simeq \mathrm{H}_0(\widetilde{X}; \mathbf{Z}) \simeq \mathbf{Z}$), so that $\Sigma^n L$ is connected.

We now observe that the following conditions are equivalent for an integer $1 \leq p \leq \frac{n}{2}$:

(*a*) The map $f : M \to X$ is p-connected.

(*b*) The map $\widetilde{f} : \widetilde{M} \to \widetilde{X}$ is p-connected.

(*c*) The spectrum $\Sigma^n V$ is p-connective.

(*d*) The spectrum $\Sigma^n L$ is p-connective.

The equivalence of (*a*) and (*b*) follows from the fact that f and $\widetilde{f}$ have the same homotopy fibers. The equivalence of (*b*) and (*c*) follows from the homotopy equivalence $C_*(\widetilde{M}; \mathbf{Z}) \simeq C_*(\widetilde{X}; \mathbf{Z}) \oplus \Sigma^n V$. To prove that (*c*) $\Rightarrow$ (*d*), we note that there is a fiber sequence

$$\Sigma^n L \to \Sigma^n V \to \mathbb{D}L.$$

The homotopy groups $\pi_i \mathbb{D}(L) \simeq \pi_i \Sigma^{n+1}(W)$ vanish for $i < \frac{n}{2}$, so that $\pi_i \Sigma^n L \to \pi_i \Sigma^n V$ is bijective for $i < \frac{n}{2}$. This proves (*c*) $\Leftrightarrow$ (*d*).

Suppose that there exists an integer $p < k - 1$ such that $\pi_p \Sigma^n L \neq 0$. Choose p as small as possible, so that $\Sigma^n L$ is p-connective. Any choice of element in $\pi_p \Sigma^n L = \pi_{p-n} L$ determines a map $\Sigma^{p-n}\mathbf{Z}[G] \to L$.

Composing with the map $L \to V$, we obtain a map $u : \Sigma^{p-n}\mathbf{Z}[G] \to V$ and a nullhomotopy of $q|\Sigma^{p-n}\mathbf{Z}[G]$. According to Theorem 4, we can lift this data to normal surgery datum $\alpha : S^p \times D^{n-q} \hookrightarrow M$. Let $f' : M' \to X$ be the normal map obtained by surgery along α, and let (V', q') be the corresponding representative for $\sigma^{vq}_{f'}$. Then (V', q') is obtained from (algebraic) surgery on V along u. It follows that L determines a Lagrangian L' in V', where L' is the cofiber of the map $\Sigma^{p-n}\mathbf{Z}[G] \to L$. Since L is finitely presented as a $\mathbf{Z}[G]$-module spectrum, its bottom homotopy group is finitely generated as a discrete $\mathbf{Z}[G]$-module. Consequently, after finitely many application of this procedure, we can reduce to the case where $\pi_p\Sigma^n L \simeq 0$: that is, where $\Sigma^n L$ is $p+1$-connective.

Applying the above argument finitely many times, we may reduce to the case where $\pi_p\Sigma^n L \simeq \pi_{p-n}L$ vanishes for $p < k-1$. Consequently, we see that $\Sigma^n L$ is $(k-1)$-connective and has projective amplitude $\leq k$. Since L is finitely presented, we can argue as in the proof of the π-π theorem to deduce that there is a fiber sequence

$$(\Sigma^{k-1}\mathbf{Z}[G])^m \xrightarrow{\phi} \Sigma^n L \to (\Sigma^k\mathbf{Z}[G])^{m'}.$$

for some integers m and m'. If $m > 0$, then the restriction of ϕ to a summand of $(\Sigma^{k-1}\mathbf{Z}[G])^m$ yields a map $\Sigma^{p-n}\mathbf{Z}[G] \to L$, where $p = k-1$. We therefore obtain a composite map $\Sigma^{p-n}\mathbf{Z}[G] \to L \to V$ and a nullhomotopy of $q|\Sigma^{p-n}\mathbf{Z}[G]$. Invoking Theorem 4, we can lift this to a normal surgery datum. Performing surgery along this datum (and replacing M by the result), we can reduce to the case where there is a fiber sequence

$$(\Sigma^{k-1}\mathbf{Z}[G])^{m-1} \xrightarrow{\phi} \Sigma^n L \to (\Sigma^k\mathbf{Z}[G])^{m'}.$$

Applying this procedure finitely many times, we reduce to the case $m = 0$: that is, $\Sigma^n L \simeq (\Sigma^k\mathbf{Z}[G])^{m'}$. If $m' > 0$, we can restrict the map $L \to V$ to a summand of L to obtain a map $\Sigma^{k-n}\mathbf{Z}[G] \to V$ and a nullhomotopy of $q|\Sigma^{k-n}\mathbf{Z}[G]$. Invoking Theorem 4 again, we can perform surgery to reduce to the case $\Sigma^n L \simeq (\Sigma^k\mathbf{Z}[G])^{m'-1}$. Applying this procedure finitely many times, we reduce to the case $L \simeq 0$. The fiber sequence

$$L \to V \to \Sigma^{-n}\mathbb{D}L$$

shows that $V \simeq 0$, so that the map $\widetilde{f} : \widetilde{M} \to \widetilde{X}$ induces an isomorphism on homology. Since $\widetilde{M}$ and $\widetilde{X}$ are simply connected, we deduce that $\widetilde{f}$ is a homotopy equivalence, so that $f : M \to X$ is also a homotopy equivalence. This completes the proof of Theorem 1 (modulo Theorem 4).