Surgery (Lecture 32)

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Our goal today is to begin the proof of the following:

Theorem 1. *Let X be a Poincare pair of dimension $n \geq 5$, ζ a stable PL bundle on X, and $f : M \to X$ a degree one normal map, where M is a PL manifold. Let $\sigma_f^{vq} \in \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$ be the relative signature of f, and suppose we are given a path p from σ_f^{vq} to the base point of $\Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$. (We can identify such a path with a Lagrangian in the Poincare object representing σ_f^{vq}, which is well-defined up to bordism). Then there exists a Δ^1-family of degree one normal maps $F : B \to X \times \Delta^1$, where B is a bordism from $M = F^{-1}(X \times \{0\})$ to a PL manifold $N = F^{-1}(X \times \{1\})$ such that F induces a homotopy equivalence $f' : N \to X$. Moreover, we can arrange that F determines a path from σ_f^{vq} to $\sigma_{f'}^{vq} = 0$ which is homotopic to p.*

Remark 2. In the last lecture, we sketched the formulation of a more general version of Theorem 1, where we replace X by a Poincare pair $(X, \partial X)$ where ∂X is already a PL manifold. To simplify the discussion, we will restrict our attention to the case where $\partial X = \emptyset$, but the ideas introduced in this lecture generalize to the relative case.

To prove Theorem 1, we need a method for producing bordisms between PL manifolds. For this, we will use the method of *surgery*. Fix a PL manifold M of dimension n. Write $n = p + q + 1$. Let D^{p+1} and D^{q+1} denote PL disks of dimension $p + 1$ and $q + 1$, respectively. Let S^p and S^q denote their boundaries (spheres of dimension p and q, respectively.

Definition 3. A *p-surgery datum* on M is a PL embedding $\alpha : S^p \times D^{q+1} \to M$.

To a first approximation, a p-surgery datum α on M is given by an embedding of PL manifolds $\alpha_0 : S^p \hookrightarrow M$ (given by restricting α to the product of S^p with the center of D^{q+1}). To obtain a surgery datum from α_0, we must additionally specify that α_0 extends to a PL homeomorphism between $S^p \times D^{q+1}$ and a neighborhood of the image of α_0. Such a homeomorphism determines a smooth structure on M along the image of α_0, with respect to which α_0 is a smooth embedding with trivialized normal bundle. Conversely, suppose we are given an embedding $\alpha_0 : S^p \to M$ and a smoothing of M along the image of α_0, such that α_0 is a smooth map. Then α_0 has a normal bundle N_{α_0}, and there is a neighborhood of $\alpha_0(S^p)$ in M which is diffeomorphic to the unit sphere bundle of N_{α_0}. In particular, if N_{α_0} is trivial, we obtain a diffeomorphism (and therefore a PL homeomorphism) of a neighborhood of $\alpha_0(S^p)$ with $S^p \times D^{q+1}$. This argument shows that we can identify a p-surgery datum on M with three pieces of data:

(i) A PL embedding $\alpha_0 : S^p \to M$.

(ii) A smoothing of M along the image of α_0 (with respect to which α_0 is a smooth map).

(iii) A trivialization of the normal bundle to α_0 (as a vector bundle).

Construction 4. Let M be a PL manifold of dimension $n = p + q + 1$ and let $\alpha : S^p \times D^{q+1} \hookrightarrow M$ be a p-surgery datum. We let $B(\alpha)$ denote the polyhedron given by

$$(M \times [0, 1]) \coprod_{\{1\} \times S^p \times D^{q+1}} (D^{p+1} \times D^{q+1}).$$

Then $B(\alpha)$ is a PL manifold with boundary, given by the disjoint union of $M \times \{0\}$ and

$$N = M - (S^p \times (D^{q+1})^{\circ}) \coprod_{S^p \times S^q} (D^{p+1} \times S^q).$$

We refer to N as the *PL manifold obtained from M via surgery along α*, and to $B(\alpha)$ as the *trace of the surgery*.

More informally: N is the manifold obtained from M by removing the interior of $S^p \times D^{q+1}$ (thereby creating a manifold with boundary $S^p \times S^q$) and gluing on a copy of $D^{p+1} \times S^q$.

Remark 5. Let N be a PL manifold obtained from surgery on a PL manifold M along a map $\alpha : S^p \times D^{q+1} \hookrightarrow M$. Then there is an evident embedding $\beta : D^{p+1} \times S^q \to N$, which is a q-surgery datum in N. Performing surgery on N along β recovers the manifold M.

We will be interested in using surgery to construct *normal bordisms* between normal maps to a Poincare complex. For this, we need a slight variation on Definition 3. Let M be a PL manifold, so that the stable normal bundle of M is classifies by a map $\chi : M \to \mathbf{Z} \times \mathrm{BPL}$. If we are given a p-surgery datum $\alpha : S^p \times D^{q+1} \to M$, then $\chi \circ \alpha$ extends canonically to a map $\gamma : D^{p+1} \times D^{q+1} \to \mathbf{Z} \times \mathrm{BPL}$.

Suppose now that X is a space equipped with a stable PL bundle ζ, and that we are given a normal map $f : M \to X$. Then ζ is classified by a map $\chi_X : X \to \mathbf{Z} \times \mathrm{BPL}$, and the normal structure on f gives a homotopy $h_0 : \chi \simeq \chi_X \circ f$.

Definition 6. In the situation above, a *normal p-surgery datum* on M consists of the following data:

(i) A p-surgery datum $\alpha : S^p \times D^{q+1} \to M$.

(ii) A map $\beta : D^{p+1} \times D^{q+1} \to X$ extending $f \circ \alpha$.

(iii) A homotopy h from $\chi_X \circ \beta$ to γ, extending the homotopy determined by h.

Given a normal p-surgery datum, we can use α to construct a bordism $B(\alpha)$ from M to a PL manifold N, β to construct a map $F : B(\alpha) \to X$ extending $f : M \to X$, and h to endow F with the structure of a Δ^1-family of normal maps.

Remark 7. Let us think of a p-surgery datum on a PL manifold M as an embedding $\alpha_0 : S^p \to M$, together with a choice of trivial normal bundle to α_0. If $f : M \to X$ is a degree one normal map, then to obtain a normal p-surgery datum we need to choose a nullhomotopy of the composite map $(f \circ \alpha_0) : S^p \to X$, which is *compatible* with the nullhomotopy of the map

$$S^p \stackrel{\alpha_0}{\to} M \stackrel{f}{\to} X \to \mathbf{Z} \times \mathrm{BPL}$$

determined by the choice of trivial normal bundle.

Let us now see what surgery can do for us in low degrees. Assume that X is a Poincare space of dimension $n \geq 5$, ζ a stable PL bundle on X, and $f : M \to X$ is a degree one normal map.

Let us begin by doing surgery in the case $p = -1$. In this case, S^p is empty and therefore a surgery datum $\alpha : S^p \times D^{q+1} \to M$ is unique. To promote α to a normal surgery datum, we need to choose a map $\beta : D^{n+1} \to X$ (up to homotopy, this a point $x \in X$), together with a trivialization of $\beta^* \zeta$. Unwinding the definitions, we see that $B(\alpha)$ is the disjoint union $(M \times [0,1]) \coprod D^{n+1}$, regarded as a bordism from M to $M \coprod S^n$. If we have chosen β and the trivialization of $\beta^* \zeta$, then we can regard this as a normal bordism from f to a map $M \coprod S^n \to X$, whose restriction to S^n is determined by β. By performing surgeries of this type, we can always arrange that the map $M \to X$ is surjective on connected components.

Now suppose that $f : M \to X$ fails to be *injective* on connected components. Then we can choose two points $x, y \in M$ belonging to different components of M and a path joining $f(x)$ to $f(y)$. Choosing small disks around the points x and y, we obtain a 0-surgery datum $\alpha : S^0 \times D^n \hookrightarrow M$. A choice of path p from $f(x)$ to $f(y)$ determines the datum (ii) required by Definition 6. We cannot always extend α to a normal surgery datum: our choice of disks determines trivializations of the fibers $\zeta_{f(x)}$ and $\zeta_{f(y)}$, which may or may not extend to a trivalization of ζ over the path p. However, the obstruction is slight by virtue of the following (non-obvious!) fact:

Claim 8. The fundamental group $\pi_1(\mathbf{Z} \times \mathrm{BPL})$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$. In other words, every orientation-preserving PL automorphism of $\mathbb{R}^n$ is isotopic to the identity, for $n \gg 0$.

In fact, more is true: the map $\pi_i(\mathbf{Z} \times \mathrm{BO}) \to \pi_i(\mathbf{Z} \times \mathrm{BPL})$ induces an isomorphism for $i \leq 6$ and a surjection when $i = 7$ (using smoothing theory, this is equivalent to the assertion that there are no exotic smooth structures on piecewise linear spheres of dimensions ≤ 6). In this lecture, we will need something much weaker: namely, that the above map is bijective for $i \leq 1$ and surjective for $i \leq 2$. Using smoothing theory, this is equivalent to the (reasonably obvious) claim that there are no exotic smooth structures on spheres of dimension ≤ 1.

In our situation, we cannot necessarily extend an *arbitrary* $\alpha : S^0 \times D^n \hookrightarrow M$ to a normal surgery datum. However, we always do so after modifying α by applying an orientation-reversing automorphism to one of the disks D^n. After making this modification, we obtain a normal bordism from M to a PL manifold with fewer connected components. Applying this procedure finitely many tiimes, we may replace $f : M \to X$ by a degree one normal map which induces an isomorphism $\pi_0 M \to \pi_0 X$.

Let us now assume that X and M are connected, and choose a base point $x \in M$. Suppose that the map $\pi_1 M \to \pi_1 X$ is not surjective. Choose another point y in M and a path q from y to x. Choose any class γ in $\pi_1 X$, and a path p from $f(x)$ to $f(y)$ such that the loop composing p with $f(q)$ represents γ. Choosing small disks around x and y, we obtain a surgery datum $\alpha : S^0 \times D^n \hookrightarrow M$ as before. The path p supplies the datum (ii) required by Definition 6, and we can argue as before (modifying α if necessary) to obtain the datum (iii). Let N be obtained from M by normal surgery along α. Since $n \geq 3$, deleting small disks around x and y does not change the fundamental group of M. Using van Kampen's theorem, we compute that $\pi_1 N$ is obtained from $\pi_1 M$ by freely adjoining an additional generator, and the map $\pi_1 N \to \pi_1 X$ carries this generator to γ (here we are being sloppy about base points here). Since X is a finite complex, its fundamental group is finitely generated. We may therefore perform this procedure finitely many times to reduce to the situation where the degree one normal map $f : M \to X$ induces a surjection $\pi_1 M \to \pi_1 X$.

Now suppose that $\pi_1 M \to \pi_1 X$ fails to be injective. Choose an element of $\pi_1 M$ whose image in $\pi_1 X$ is trivial. We can represent this element by a map $\alpha_0 : S^1 \to M$. Since the dimension of M is ≥ 3, a general position argument allows us to assume that α_0 is an embedding. The composite map $S^1 \to M \to X$ is nullhomotopic, so that the stable normal bundle of M is trivial in a neighborhood of α_0 and we may therefore assume that M is smooth in a neighborhood of α_0. The normal bundle to α_0 is stable trivial, hence orientable and therefore trivial. We may therefore extend α_0 to an embedding $\alpha : S^1 \times D^{n-1} \hookrightarrow M$. Choose a nullhomotopy of $f \circ \alpha$. As before, it is not clear that we can choose datum (iii) required by Definition 6: we encounter an obstruction in $\pi_2(\mathbf{Z} \times \mathrm{BPL})$. However, since the map $\pi_2(\mathbf{Z} \times \mathrm{BO}) \to \pi_2(\mathbf{Z} \times \mathrm{BPL})$ is surjective, we can adjust our original embedding α (choosing a different trivialization of the normal bundle to α_0) to make this obstruction vanish. This allows us to perform a normal surgery on the manifold M, thereby obtaining a cobordant degree one normal map $f' : N \to X$. Since the dimension of M is ≥ 4, removing a neighborhood of $\alpha_0(S^1)$ does not change the fundamental group of M. Consequently, we can use van Kampen's theorem to compute the fundamental group of N: it is obtained from the fundamental group of M by killing the normal subgroup generated by γ.

Since X is a finite complex, the fundamental group $\pi_1 X$ is finitely presented. Since $\pi_1 M$ is finitely generated, the surjective map $\pi_1 M \to \pi_1 X$ exhibits $\pi_1 X$ as the quotient of $\pi_1 M$ by the normal subgroup generated by finitely many elements of $\pi_1 M$. It follows that, after a finite number of applications of the above procedure, we may replace $f : M \to X$ by a degree one normal map which induces an isomorphism of fundamental groups.