The Main Theorem: First Reduction (Lecture 31)

April 12, 2011

Let X be a Poincare space of dimension ≥ 5. Our goal for the next few lectures is to sketch a proof of the main theorem of surgery theory: there is a fiber sequence of spaces

$$\mathbb{S}(X) \to \mathbb{S}^n(X) \to \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X).$$

In other words, we have a diagram

$$\begin{array}{ccc} \mathbb{S}(X) & \xrightarrow{\phi} & \mathbb{S}^n(X) \\ \downarrow & & \downarrow \\ * & \xrightarrow{\phi'} & \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X) \end{array}$$

and we wish to show that it is a homotopy pullback square.

Fix a point of $\mathbb{S}^n(X)$, corresponding to a PL reduction ζ of the Spivak bundle of X and a degree one normal map $f_0 : M_0 \to X$. We will abuse notation by simply referring to f as a point of $\mathbb{S}^n(X)$. Let $\sigma^{vq}_{f_0}$ denote the image of f_0 in $\mathbb{L}^{vq}(X, \zeta_X)$. We wish to prove that the above diagram induces a homotopy equivalence from the homotopy fiber of ϕ over f to the homotopy fiber of ϕ' over $\sigma^{vq}_{f_0}$.

We now wish to describe the homotopy fiber of ϕ in more explicit geometric terms. Recall that ϕ is given as the geometric realization of a map of simplicial spaces $\phi_\bullet : \mathbb{S}(X)_\bullet \to \mathbb{S}^n(X)_\bullet$. Unfortunately, this map is not a Kan fibration, so the homotopy fiber cannot be computed "pointwise". We will therefore need to make an auxiliary construction.

We define a simplicial space $P_\bullet$ so that P_k is the homotopy fiber product $\{f_0\} \times_{\mathbb{S}^n(X)_0} \mathbb{S}^n(X)_{k+1}$. More informally, P_k is a classifying space for families of degree one normal maps $f : M \to X \times \Delta^{k+1}$ such that the fiber of f over the vertex $\{0\} \in \Delta^{k+1}$ is the normal map $f_0 : M_0 \to X$.

Restricting to the faces opposite the 0th vertex, we obtain a map of simplicial spaces $P_\bullet \to \mathbb{S}^n(X)_\bullet$, which is easily seen to be a Kan fibration. One can also see that the map $P_\bullet \to \{f_0\}$ is a trivial Kan fibration, so that the geometric realization $|P_\bullet|$ is contractible. This realization is a model for the space of paths in $\mathbb{S}^n(X)$ beginning at the point f_0.

The homotopy fiber of ϕ over f_0 is given by the fiber product

$$\{f_0\} \times_{\mathbb{S}^n(X)} \mathbb{S}(X) \simeq |P_\bullet| \times_{|\mathbb{S}^n(X)_\bullet|} |\mathbb{S}(X)_\bullet|.$$

Since the map $P_\bullet \to \mathbb{S}^n(X)_\bullet$ is a Kan fibration, we can compute this as $|U_\bullet|$, where $U_k \simeq P_k \times_{\mathbb{S}^n(X)_k} \mathbb{S}(X)_k$. More explicitly, U_k is a classifying space for degree one normal maps $f : M \to X \times \Delta^{k+1}$ for which the fiber over $\{0\} \subseteq \Delta^{k+1}$ is $f_0 : M_0 \to X$, and f induces a homotopy equivalence $f^{-1}(X \times \tau) \to X \times \tau$ for every simplex τ which does not contain the vertex 0.

Let P' be the space of paths in $\Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$ from $\sigma^{vq}_{f_0}$ to the base point, and let $P'_\bullet$ be the constant simplicial space taking the value P'. We have a map of simplicial spaces

$$\psi : U_\bullet \to P'_\bullet.$$

To prove the main theorem, it will suffice to show that this map induces a homotopy equivalence of geometric realizations. In fact, we claim that ψ is a trivial Kan fibration of simplicial spaces. In other words, we claim that for each $k \geq 0$, the canonical map

$$\theta_k : U_k \to U_\bullet(\partial\,\Delta^k) \times_{\mathrm{Map}(\partial\,\Delta^k, P)} \mathrm{Map}(\Delta^k, P)$$

is surjective on connected components.

Let us spell out this out more explicitly in the special case $k = 0$. The space U_0 classifies degree one maps $f : M \to X \times \Delta^1$ such that $f^{-1}(X \times \{0\}) \simeq M_0$ and $M_1 = f^{-1}(X \times \{1\})$ is homotopy equivalent to X. In other words, U_0 is a classifying space for *normal bordisms* from M_0 to a manifold homotopy equivalent to X. Every such bordism determines a Lagrangian in the Poincare object representing $\sigma^{vq}_{f_0}$, which we can regard as a path from $\sigma^{vq}_{f_0}$ to the base point in $\Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$. This path is the image of M under the map θ_0. Consequently, when $k = 0$, we wish to prove the following:

Theorem 1. *Let X be a Poincare space of dimension $n \geq 5$ equipped with a PL reduction of its tangent bundle, let $f_0 : M_0 \to X$ be a degree one normal map determining a Poincare object $\sigma^{vq}_{f_0}$ of $(\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{Sp})^{\mathrm{fp}}, Q^q_{\zeta_X})$, and let L be a Lagrangian in $\sigma^{vq}_{f_0}$. Then L is cobordant to a Lagrangian arising from a normal bordism from $f_0 : M_0 \to X$ to a homotopy equivalence $f_1 : M_1 \to X$.*

Let us now extend our analysis to the case where k is arbitrary. Let $D \subseteq \Delta^{k+1}$ be the subset obtained by deleting the interior and the face opposite the 0th vertex. Then D is a PL disk equipped with preferred triangulation. Every point of $U_\bullet(\partial\,\Delta^k)$ determines D-family of normal maps $f : M \to X \times D$ such that the fiber over the vertex $\{0\} \subseteq \Delta^{k+1}$ (which we can think of as the "center" of the disk D) is given by M_0, and $f^{-1}(X \times \tau)$ is homotopy equivalent to $X \times \tau$ for every simplex $\tau \subseteq \partial\,D$. Such data determines a map $u : D \to \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$ which is trivial on $\partial\,D$ (and carries the center of D to the point $\sigma^{vq}_{f_0}$, but this is not important in what follows). We may think of u as a point of $\Omega^{\infty+k}\mathbb{L}^{vq}(X, \zeta_X)$, which we can represent by Poincare object $(\mathcal{F}, q)$ of $(\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{Sp}^{\mathrm{fp}}, \Omega^k \zeta_X)$.

Lifting f to a point of U_k amounts to extending f to a Δ^{k+1}-family of normal maps $\overline{f} : \overline{M} \to X \times \Delta^{k+1}$, such that $\overline{f}^{-1}(X \times \tau)$ is homotopy equivalent to $X \times \tau$ for every face $\tau \subseteq \Delta^{k+1}$ not containing the 0th vertex. Such an extension can be described by first specifying the inverse image of the face of Δ^{k+1} opposite the 0th vertex: this is a PL manifold N equipped with a homotopy equivalence $g : N \to X \times \Delta^k$ (neat over Δ^k) such that $g^{-1}(X \times \partial\,\Delta^k)$ is PL homeomorphic to $f^{-1}(X \times \partial\,D)$, together with a normal bordism from M to N which is compatible with the PL homeomorphism on the boundaries.

Such a lifting determines an extension of σ^{vq}_f to a map $\sigma^{vq}_{\overline{f}} : \Delta^{k+1} \to \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$ which vanishes on the face of Δ^{k+1} opposite the 0th vertex. In particular, we obtain a Lagrangian in the Poincare object $(\mathcal{F}, q)$, which is well-defined up to bordism. Unwinding the definitions, the surjectivity of θ_k amounts to the following assertion: every Lagrangian in $(\mathcal{F}, q)$ arises (up to cobordism) from a normal bordism from M to N as in the above paragraph.

It is convenient to rephrase the above construction by replacing the Poincare space X by the Poincare pair $(X \times D, X \times \partial\,D)$. Note that the Spivak bundle $\zeta_{X \times D}$ is given by $p^*(\Omega^k \zeta_X)$, where p denotes the projection $X \times D \to X$.

Let now embark on a brief digression about how some of the constructions of the last few lectures generalize to the setting of Poincare pairs. Our definition of a degree one structure on a map of Poincare spaces $f : Y \to X$ generalizes to the setting of a map of Poincare pairs $f : (Y, \partial\,Y) \to (X, \partial\,X)$. To such a map, one can assign a signature

$$\sigma^{vq}_f : S \to \mathbb{L}^{vq}(X, \partial\,X, \zeta_X) = \mathrm{cofib}(\mathbb{L}^{vq}(\partial\,X, \zeta_X|\,\partial\,X) \to \mathbb{L}^{vq}(X, \zeta_X).$$

We have a fiber sequence of spectra

$$\mathbb{L}^{vq}(X, \zeta_X) \to \mathbb{L}^{vq}(X, \partial\,X, \zeta_X) \to \mathbb{L}^{vq}(\partial\,X, \zeta_{\partial\,X}),$$

where the second map carries σ_f^{vq} to $\sigma^{vq}\partial f$, where $\partial f : \partial Y \to \partial X$ denotes the induced degree one map of Poincare spaces. In the special case where ∂f is a homotopy equivalence, we obtain a canonical lifting of σ_f^{vq} to a point of $\Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$.

Let us now return to the problem of showing that the map

$$\theta_k : U_k \to U_\bullet(\partial \Delta^k) \times_{\mathrm{Map}(\partial \Delta^k, P)} \mathrm{Map}(\Delta^k, P)$$

is surjective. A point of the codomain determines a degree one normal map of Poincare pairs $f : (M, \partial M) \to (X \times D, X \times \partial D)$ which is a homotopy equivalence on the boundary, The relative signature σ_f^{vq} can be identified with the map u under the homotopy equivalence $\mathbb{L}^{vq}(X \times D, \zeta_{X \times D}) \simeq \mathbb{L}^{vq}(X, \zeta_X)$. Consequently, verifying the surjectivity of θ_k amounts to verifying that Theorem 2 is satisfied, where we replace the Poincare space X by the Poincare pair $(X \times D, X \times \partial D)$. This is in turn a special case of the following more general assertion:

Theorem 2. *Let $(X, \partial X)$ be a Poincare pair of dimension $n \geq 5$ equipped with a PL reduction of its Spivak bundle, let $f : (M, \partial M) \to (X, \partial X)$ be a degree one normal map inducing a homotopy equivalence $\partial M \to \partial X$, and let L be a Lagrangian in Poincare object representing σ_f^{vq}. Then L is cobordant to a Lagrangian which arises from a normal bordism (constant along ∂M) from M to a homotopy equivalence $g : (N, \partial M) \to (X, \partial X)$.*

Remark 3. Using Theorem 2, we can extend the statement of the main theorem of the previous lecture to the case of Poincare pairs. That is, if $(X, \partial X)$ is a Poincare pair of dimension $n \geq 5$ where ∂X is a PL manifold of dimension $n - 1$, then we have a fiber sequence

$$\mathbb{S}(X) \to \mathbb{S}^n(X) \to \mathbb{L}^{vq}(X, \zeta_X).$$