Statement of the Main Theorem (Lecture 30)

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Our first goal in this lecture is to complete the analysis of the previous lecture, constructing a quadratic refinement of difference $f_!(\sigma_Y^{vs}) - \sigma_X^{vs}$ when $f : Y \to X$ is a map of Poincare spaces with a degree one structure. Recall that we need to show the following:

Proposition 1. *Let Z be a space, and suppose we are given a splitting of the canonical map of spectra $\Sigma^\infty_+ Z \to \Sigma^\infty_+ * \simeq S$, giving a map of spectra $e : S \to \Sigma^\infty_+ Z$ and a decomposition of spectra $\Sigma^\infty_+ Z \simeq E \oplus S$. Then the composite map*

$$q_- : S \xrightarrow{e} \Sigma^\infty_+ Z \to (\Sigma^\infty_+ Z \wedge \Sigma^\infty_+ Z)^{h\Sigma_2} \to (E \wedge E)^{h\Sigma_2}$$

factors canonically through the transfer map $\mathrm{tr} : (E \wedge E)_{h\Sigma_2} \to (E \wedge E)^{h\Sigma_2}$.

In other words, we claim that there is a canonical nullhomotopy of the composite map

$$S \xrightarrow{e} \Sigma^\infty_+ Z \to (\Sigma^\infty_+ Z \wedge \Sigma^\infty_+ Z)^{h\Sigma_2} \to (E \wedge E)^{h\Sigma_2} \to (E \wedge E)^{t\Sigma_2}.$$

For every spectrum M, let $T(M) = (M \wedge M)^{t\Sigma_2}$. Then $T(M)$ is an exact functor. If M is a finite spectrum, we can write $T(M) = M \wedge T(S) = M \wedge S^{t\Sigma_2}$. More generally, we obtain a map $\alpha_M : M \wedge S^{t\Sigma_2} \to T(M)$, which need not be a homotopy equivalence when M is not finite.

Let Z be a space and $M = \Sigma^\infty_+ Z$. The diagonal $Z \to Z \times Z$ induces a map of spectra $M \to (M \wedge M)^{h\Sigma_2}$. Composing with the map $(M \wedge M)^{h\Sigma_2} \to (M \wedge M)^{t\Sigma_2}$, we obtain a map $\beta_Z : M \to T(M)$, depending functorially on Z. Taking Z to be a point, we obtain a map $\beta_0 : S \to T(S)$. Note that β_0 induces a map

$$\gamma : M = C_*(Z; \underline{S}) \to C_*(Z; \underline{T(S)}) = M \wedge T(S).$$

By naturality, we deduce that β_Z factors as a composition

$$M \xrightarrow{\gamma} M \wedge T(S) \xrightarrow{\alpha_M} T(M).$$

We wish to show that the composite map

$$S \xrightarrow{e} M \xrightarrow{\beta_Z} T(M) \to T(E)$$

is nullhomotopic. Using the above analysis, we can write this map as a composition

$$S \xrightarrow{e} M \xrightarrow{M \wedge \beta_0} M \wedge T(S) \xrightarrow{\alpha_M} T(M) \to T(E),$$

which is also the composition

$$S \xrightarrow{e} M \to E \xrightarrow{E \wedge \beta_0} E \wedge T(S) \xrightarrow{\alpha_E} T(E).$$

Since E is defined as the cofiber of e, the composition of the first two maps is canonically nullhomotopic. This completes the proof of Proposition 1.

Let us now recall our application of Proposition 1. Let $f : Y \to X$ be a map of Poincare spaces with a degree one structure. Let $\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}$ denote the smallest stable subcategory of locally constant sheaves of spectra on X which contains $i_! S$, for every point $i : \{x\} \hookrightarrow X$ (if X is connected with base point x, so that $\mathrm{Shv}_{\mathrm{lc}}(X)$ can be identified with the ∞-category of right modules over $R : \Sigma^\infty_+ \Omega(X)$, then $\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}$ is the full subcategory spanned by the finitely presented R-modules). Let ζ_X denote the Spivak bundle of X, so that we have quadratic functors

$$Q^s_{\zeta_X}, Q^q_{\zeta_X} : \mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}} \to \mathrm{Sp}$$

given by

$$Q^s_{\zeta_X}(\mathcal{F}) = C_*(X; \zeta_X \wedge (\mathcal{F} \wedge \mathcal{F})^{h\Sigma_2}) \simeq C^*(X; (\mathcal{F} \wedge \mathcal{F})^{h\Sigma_2})$$
$$Q^q_{\zeta_X}(\mathcal{F}) = C_*(X; \zeta_X \wedge (\mathcal{F} \wedge \mathcal{F})_{h\Sigma_2}) \simeq C^*(X; (\mathcal{F} \wedge \mathcal{F})_{h\Sigma_2}).$$

Applying Proposition 1 to the homotopy fibers of the map f, we obtain a quadratic object $(\mathcal{E}, q_f)$ of $(\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}, Q^q_{\zeta_X})$, where $\mathcal{E}$ denotes the homotopy fiber of the map of local systems $f_! \underline{S} \to \underline{S}$. Let $\mathrm{tr}(q_f)$ denote the image of q_f under the transfer map $Q^q_{\zeta_X}(\mathcal{E}) \to Q^s_{\zeta_X}(\mathcal{E})$. We then have an equivalence of Poincare objects

$$(\mathcal{E}, \mathrm{tr}(q_f)) \oplus (\underline{S}, q_X) \simeq f_!(\underline{S}, q_Y)$$

where $(\underline{S}, q_X)$ represents the visible symmetric signature of X and $(\underline{S}, q_Y)$ represents the visible symmetric signature of Y. It follows that $(\mathcal{E}, \mathrm{tr}(q_f))$ is a Poincare object of $(\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}, Q^s_{\zeta_X})$, so that $(\mathcal{E}, q_f)$ is a Poincare object of $(\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}, Q^q_{\zeta_X})$. It therefore determines a point in the zeroth space of the visible quadratic L-theory spectrum $\mathbb{L}^{vq}(X, \zeta_X) = \mathbb{L}^{vq}(X; \zeta_X, S)$. We will denote this point by σ^{vq}_f.

Let us now fix the Poincare space X, and let $\mathbb{S}^n(X)_0$ denote the classifying space for stable PL bundles ζ on X and degree one normal maps $f : M \to X$, where M is a compact PL manifold. The above construction determines a map of classifying spaces

$$\mathbb{S}^n(X)_0 \to \mathrm{Poinc}(\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}, Q^q_{\zeta_X})_0.$$

More generally, given a Δ^k-family of normal maps $f : M \to X \times \Delta^k$, we can apply the above construction to each of the induced maps $f^{-1}(X \times \tau) \to X$, to obtain a Poincare object of the ∞-category $\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}_{[k]}$ (see Lecture 6). This construction is classified by a map of spaces

$$\mathbb{S}^n(X)_k \to \mathrm{Poinc}(\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}, Q^q_{\zeta_X})_k.$$

Passing to geometric realizations, we obtain a map

$$\mathbb{S}^n(X) \to \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X).$$

Note that if a degree one normal map $f : M \to X$ is actually a homotopy equivalence, then $f_! \underline{S} \to \underline{S}$ is invertible, so that the quadratic object $(\mathcal{E}, q_f)$ constructed above is trivial. It follows that the composite map

$$\mathbb{S}(X)_\bullet \to \mathbb{S}^n(X)_\bullet \to \mathrm{Poinc}(\mathrm{Shv}_{\mathrm{lc}}(X)^{\mathrm{fp}}, Q^q_{\zeta_X})_\bullet$$

is canonically nullhomotopic (as a map of simplicial spaces), so that the composite map

$$\mathbb{S}(X) \to \mathbb{S}^n(X) \to \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$$

is likewise nullhomotopic.

We are now ready to state the main theorem of this course:

Theorem 2. *Let X be a Poincare space of dimension ≥ 5. Then*

$$\mathbb{S}(X) \to \mathbb{S}^n(X) \to \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$$

is a fiber sequence. In other words, the structure space $\mathbb{S}(X)$ can be regarded as the homotopy fiber of a map $\mathbb{S}^n(X) \to \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X)$, which carries every degree one normal map $f : M \to X$ to the signature σ^{vq}_f.

Using the π-π Theorem, we can be more precise. Let us assume that X is connected. Choose a base point $x \in X$ and a trivialization $\zeta_X(x) \simeq S^{-n}$ of the Spivak fibration at the point x (so that n is the dimension of the Poincare space X). Then $\Sigma^n \zeta_X$ has a trivialization at x, so that we have

$$\mathbb{L}^{vq}(X, \zeta_X) \simeq \Sigma^{-n} \mathbb{L}^{vq}(X, \Sigma^n \zeta_X) \simeq \Sigma^{-n} \mathbb{L}^q(R) \simeq \Sigma^{-n} \mathbb{L}^q(\mathbf{Z}[\pi_1 X]),$$

where R is the A_∞-ring $\Sigma^\infty_+(\Omega X)$ and the last equivalence follows from the π-π theorem. Here $\mathbf{Z}[\pi_1 X]$ is equipped with the involution given by $\gamma \mapsto \pm\gamma^{-1}$ for $\gamma \in \pi_1 X$, where the signs are given by the obstruction to orienting the Spivak bundle ζ_X.

Given a stable PL reduction ζ of the Spivak bundle of X, we obtain a point of $\mathbb{S}^{tn}(X) \simeq \mathbb{S}^n(X)$, which determines an element γ of the abelian group $L^q_n(\mathbf{Z}[\pi_1 X])$ (represented by σ^{vq}_f, where $f : M \to X$ is a degree one normal map to (X, ζ)). This element vanishes if we can choose f to be a homotopy equivalence. Theorem 2 asserts the converse: if the dimension n is greater than 5 and γ vanishes, then X is homotopy equivalent to a PL manifold M (having stable normal bundle given by the pullback of ζ).