Stable ∞-Categories (Lecture 3)

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In the last lecture, we introduced the definition of an *∞-category* as a generalization of the usual notion of category. This definition is one way of formalizing the notion of a higher category in which all k-morphisms are invertible for $k > 1$. Many other approaches are possible. The following is probably more intuitive:

Definition 1. A *topological category* is a category $\mathcal{C}$ together with a topology on the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for every pair of objects X and Y, such that the composition maps $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ are continuous. (In other words, a category which is *enriched* over the category of topological spaces.) If $\mathcal{C}$ is a topological category containing a pair of objects X and Y, we will denote $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ by $\mathrm{Map}_{\mathcal{C}}(X, Y)$ when we wish to emphasize that we are thinking of it as a topological space.

Remark 2. To accommodate certain examples, it is convenient to modify Definition 1 by working with *compactly generated* topological spaces rather than topological spaces. That is, we require that each $\mathrm{Map}_{\mathcal{C}}(X, Y)$ be compactly generated, and require that composition is given by continuous maps

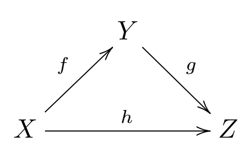
$$\mathrm{Map}_{\mathcal{C}}(X, Y) \times \mathrm{Map}_{\mathcal{C}}(Y, Z) \to \mathrm{Map}_{\mathcal{C}}(X, Z)$$

where the product is taken in the category of compactly generated topological spaces. This is a technical point which may be safely ignored.

The theory of ∞-categories is closely related to the theory of topological categories.

Construction 3 (Sketch)**.** Let $\mathcal{C}$ be a topological category. We define a simplicial set $\mathrm{N}^t(\mathcal{C})$, the *homotopy coherent nerve* of $\mathcal{C}$, as follows:

- The 0-simplices of $\mathrm{N}^t(\mathcal{C})$ are the objects of $\mathcal{C}$.
- The 1-simplices of $\mathrm{N}^t(\mathcal{C})$ are morphisms $f : X \to Y$ in $\mathcal{C}$.
- The 2-simplices of $\mathrm{N}^t(\mathcal{C})$ are given by (noncommuting) diagrams



  in $\mathcal{C}$, together with a choice of path from h to $g \circ f$ in $\mathrm{Map}_{\mathcal{C}}(X, Z)$.
- $\cdots$

Example 4. Let $\mathcal{C}$ be an ordinary category. We can regard $\mathcal{C}$ as a topological category by endowing each mapping set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ with the discrete topology.

It turns out that for any topological category $\mathcal{C}$, the homotopy coherent nerve $\mathrm{N}^t(\mathcal{C})$ is an ∞-category. Moreover, there is a sort of converse: every ∞-category is equivalent to $\mathrm{N}^t(\mathcal{C})$ for some topological category $\mathcal{C}$, and the topological category $\mathcal{C}$ is essentially unique (up to a suitable notion of weak homotopy equivalence). In other words, the construction $\mathcal{C} \mapsto \mathrm{N}^t(\mathcal{C})$ determines an equivalence between the theory of topological categories and the theory of ∞-categories. From this point forward, we will work at an informal level and freely mix these two notions. For example, if S is an ∞-category containing a pair of 0-simplices x and y, we will use $\mathrm{Map}_S(x, y)$ to denote a mapping space between x and y, when viewed as objects of a topological category whose homotopy coherent nerve is equivalent to S.

Remark 5. To any topological category $\mathcal{C}$ (and therefore to any ∞-category) we can associate an ordinary category $\mathrm{h}\mathcal{C}$, called the *homotopy category* of $\mathcal{C}$. It has the same objects, with morphisms given by $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y)$.

Example 6. The collection of finite CW complexes forms a topological category: for any pair of finite CW complexes X and Y, we can endow the set of continuous maps $\mathrm{Hom}(X, Y)$ with the compact-open topology. If we use the convention of Remark 2, then this generalizes to arbitrary CW complexes. We will denote the homotopy coherent nerve of this (larger) topological category by $\mathcal{S}$, and refer to it as the *∞-category of spaces.*

Here is an example of greater interest to us:

Example 7 (Sketch)**.** Let A be an associative ring. There is an ∞-category $\mathcal{D}^{\mathrm{perf}}(A)$ which may be described as follows:

- The 0-simplices of $\mathcal{D}^{\mathrm{perf}}(A)$ are given by bounded chain complexes of finitely generated projective left A-modules
$$\cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to \cdots$$
- A 1-simplex of $\mathcal{D}^{\mathrm{perf}}(A)$ consists of a pair of chain complexes $P_\bullet$ and $Q_\bullet$, together with a map of chain complexes $f : P_\bullet \to Q_\bullet$.
- A 2-simplex of $\mathcal{D}^{\mathrm{perf}}(A)$ consists of a (not necessarily commutative) diagram of chain complexes
$$\begin{array}{ccccc} & & Q_\bullet & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ P_\bullet & & \xrightarrow{\quad h \quad} & & R_\bullet, \end{array}$$
together with a chain homotopy from h to $g \circ f$.
- Higher dimensional simplices are defined using higher-order chain homotopies.

The homotopy category of $\mathcal{D}^{\mathrm{perf}}(A)$ is equivalent to the category **hPerf** of the previous lecture.

Our next goal is to axiomatize some of the special features enjoyed by ∞-categories of the form $\mathcal{D}^{\mathrm{perf}}(A)$. First, we need to introduce a bit of terminology. Let $\mathcal{C}$ be an ∞-category and $\mathcal{I}$ an ordinary category. It makes sense to speak of *functors* from $\mathcal{I}$ to $\mathcal{C}$: these are given by maps of simplicial sets $\mathrm{N}(\mathcal{I}) \to \mathcal{C}$. We can use this notion to make sense of commutative diagrams in $\mathcal{C}$. For example, a square diagram
$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$
in $\mathcal{C}$ is just a map of simplicial sets $\Delta^1 \times \Delta^1 \to \mathcal{C}$.

Definition 8. Let $\mathcal{C}$ be an ∞-category. We will say that an object $0 \in \mathcal{C}$ is a *zero object* if, for every object $X \in \mathcal{C}$, the mapping spaces

$$\mathrm{Map}_{\mathcal{C}}(0, X) \qquad \mathrm{Map}_{\mathcal{C}}(X, 0)$$

are contractible. We will say that $\mathcal{C}$ is *pointed* if it admits a zero object.

If $\mathcal{C}$ is a pointed ∞-category, then for any pair of objects X and Y there is a *zero map* from X to Y, given by the composition

$$X \to 0 \to Y$$

where 0 is a zero object of $\mathcal{C}$. This map is well-defined up to a contractible space of choices.

Definition 9. Let $\mathcal{C}$ be a pointed ∞-category. A *triangle* in $\mathcal{C}$ consists of a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathcal{C}$, together with a path from $g \circ f$ to the zero map in $\mathrm{Map}_{\mathcal{C}}(X, Z)$ (in other words, a *nullhomotopy* of $g \circ f$). More formally: a triangle in $\mathcal{C}$ is a square diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Z \end{array}$$

where 0 is a zero object of $\mathcal{C}$.

Suppose we are given a triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in a pointed ∞-category $\mathcal{C}$. We will say that this triangle is a *fiber sequence* if, for every object $C \in \mathcal{C}$, the associated sequence of topological spaces

$$\mathrm{Map}_{\mathcal{C}}(C, X) \to \mathrm{Map}_{\mathcal{C}}(C, Y) \to \mathrm{Map}_{\mathcal{C}}(C, Z)$$

is a homotopy fiber sequence. In this case, X is determined (up to equivalence) by g. We will say that X is the *fiber of* g and write $X = \mathrm{fib}(g)$.

Dually, we say that

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a *cofiber sequence* if, for every object $C \in \mathcal{C}$, the associated sequence of topological spaces

$$\mathrm{Map}_{\mathcal{C}}(Z, C) \to \mathrm{Map}_{\mathcal{C}}(Y, C) \to \mathrm{Map}_{\mathcal{C}}(X, C)$$

is a homotopy fiber sequence. In this case, Z is determined (up to equivalence) by f. We will say that Z is the *cofiber of* g and write $Z = \mathrm{cofib}(g)$.

Definition 10. Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is *stable* if the following conditions are satisfied:

(1) $\mathcal{C}$ is pointed: that is, there is a zero object of $\mathcal{C}$.

(2) Every morphism $f : X \to Y$ in $\mathcal{C}$ has a fiber and a cofiber.

(3) A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence.

Example 11. There is a stable ∞-category Sp whose objects are spectra. The homotopy category hSp is the classical *stable homotopy category.*

Example 12. The ∞-category $\mathcal{D}^{\mathrm{perf}}(A)$ considered above is stable.

Let $\mathcal{C}$ be a stable ∞-category. We define an abelian group $K_0(\mathcal{C})$ as follows: $K_0(\mathcal{C})$ is obtained from the free abelian group generated by symbols $[X]$, where X is an object of $\mathcal{C}$, subject to the following relation: if there is a fiber sequence

$$X \to Y \to Z,$$

then $[Y] = [X] + [Z]$.

Remark 13. Let $\mathcal{C}$ be a stable ∞-category. One can show that the homotopy category of $\mathcal{C}$ is triangulated. Moreover, the K-group $K_0(\mathcal{C})$ depends only on the homotopy category of $\mathcal{C}$, viewed as a triangulated category. However, when discussing more sophisticated invariants of $\mathcal{C}$ (like higher K-groups or L-groups) it is better not to pass to the homotopy category.

Example 14. If A is a ring, then the K-group $K_0(\mathcal{D}^{\mathrm{perf}}(A))$ is canonically isomorphic to the group $K_0(A)$ defined in the last lecture. In particular, if A is a field, then there is a canonical isomorphism $K_0(\mathcal{D}^{\mathrm{perf}}(A)) \simeq \mathbf{Z}$. If $P_\bullet$ is an object of $\mathcal{D}^{\mathrm{perf}}(A)$, then $[P_\bullet] \in K_0(\mathcal{D}^{\mathrm{perf}}(A)) \simeq \mathbf{Z}$ can be identified with the Euler characteristic

$$\sum_i (-1)^i \dim_A \mathrm{H}_i(P_\bullet)$$

of the chain complex $P_\bullet$.

In general, if we are given an object X of a stable ∞-category $\mathcal{C}$, then we can view the class $[X] \in K_0(\mathcal{C})$ as a kind of "generalized Euler characteristic" of X. It is in general not an integer, but an element of the abelian group $K_0(\mathcal{C})$ which depends on $\mathcal{C}$. We can think of the construction $\mathcal{C} \mapsto K_0(\mathcal{C})$ as a kind of *categorificatied* Euler characteristic:

Input	Invariant
vector space V	dimension $\dim(V) \in \mathbf{Z}$
complex of vector spaces	Euler characteristic $\chi \in \mathbf{Z}$
stable ∞-category $\mathcal{C}$	abelian group $K_0(\mathcal{C})$
object of a stable ∞-category $\mathcal{C}$	K-theory class $[X] \in K_0(\mathcal{C})$

However, these are not the invariants we really want to study in this class. Recall that if V is a finite dimensional vector space over $\mathbb{R}$ with a nondegenerate quadratic form q, then V is determined (up to isometry) by two invariants: the dimension of V and the signature of V. It is the latter that we would really like to generalize. Let us briefly indicate the form that this generalization will take:

Input	Invariant
nondegenerate quadratic space (V, q) over $\mathbb{R}$	signature $\sigma(V, q) \in \mathbf{Z}$
Poincare chain complex (V_*, q) over $\mathbb{R}$	signature σ of middle homology
stable ∞-category $\mathcal{C}$ with quadratic functor Q	L-group $L_0(\mathcal{C}, Q)$
Object $X \in \mathcal{C}$ satisfying Poincare duality	$[X] \in L_0(\mathcal{C}, Q)$

We will start making sense of some of these words in the next lecture.