Quadratic Refinements (Lecture 29)

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Let X be a Poincare space. Our goal is to understand the homotopy type of the piecewise linear structure space $\mathbb{S}(X)$. In the last lecture, we introduced the structure space $\mathbb{S}^n(X)$ of normal maps to X. Roughly speaking, the points of $\mathbb{S}(X)$ are compact PL manifolds M equipped with a homotopy equivalence $f : M \to X$, while the points of $\mathbb{S}^n(X)$ are degree one normal maps $f : M \to X$. Every homotopy equivalence $M \to X$ can be viewed as a degree one normal map (for an essentially unique PL reduction of the Spivak bundle of X), and this observation underlies a map of structure spaces

$$\theta : \mathbb{S}(X) \to \mathbb{S}^n(X).$$

Since the homotopy type of $\mathbb{S}^n(X)$ can be understood by means of obstruction theory, we are reduced to studying the map θ and its homotopy fibers.

We therefore ask the following question: given a PL bundle ζ on X and a normal map $f : M \to X$, how far is f from a homotopy equivalence? We would like to give an answer in terms of the formalism of L-theory. Fix an A_∞-ring R with involution, and let ζ_0 denote the underlying spherical fibration of ζ (which can be identified with the Spivak fibration of X), so that X has a visible symmetric signature $\sigma_X^{vs} \in \Omega^\infty \mathbb{L}^{vs}(X, \zeta_0, R)$. Similarly, M has a visible symmetric signature $\sigma_M^{vs} \in \Omega^\infty \mathbb{L}^{vs}(M, f^*\zeta_0, R)$. Since M is a PL manifold, this lifts canonically to a point $\sigma_M^s \in \Omega^\infty \mathbb{L}^s(M, f^*\zeta_0, R)$. The map f induces pushforward maps in L-theory, which we will denote by $f_!$. If f is a homotopy equivalence, then $f_!(\sigma_M^{vs}) \simeq \sigma_X^{vs}$ so that $f_!(\sigma_M^s)$ is a preimage of σ_X^{vs} under the assembly map. In general, this need not be the case: a degree one normal map $f : M \to X$ generally does not carry the visible symmetric signature of M to the visible symmetric signature of X.

Example 1. Suppose that we are given a degree one normal map $f : M \to S^{4k}$. The signature of M need not be zero (despite the fact that the signature of S^{4k} is zero). However, the signature of M is constrained. Since the stable normal bundle of S^{4k} is trivial, we conclude that the stable normal bundle to M is trivial. In particular, all Stiefel-Whitney classes of M are trivial, so that the Wu class of M vanishes. It follows that the intersection form on the middle dimensional homology of M is even: that is, it refines to a quadratic form, so that the signature of M must be divisible by 8.

Our goal in this lecture is to demonstrate that the phenomenon of Example 1 is quite general: if $f : M \to X$ is a degree one normal map, then the difference $f_!(\sigma_M^{vs}) - \sigma_X^{vs}$ can be canonically lifted to the *visible quadratic* L-theory space $\Omega^\infty \mathbb{L}^{vq}(X, \zeta_0, R)$.

The assertion above has nothing to do with the fact that M is actually a manifold. Let us therefore work a little bit more generally.

Definition 2. Let $f : Y \to X$ be a map of spaces, both of which have the homotopy type of a finite complex. A *degree one structure* on f consists of the following data:

(1) A spherical fibration ζ_X on X.

(2) A map of spectra $[Y] : S \to C_*(Y; f^*\zeta_X)$ satisfying the following conditions:

  (a) For every local system of spectra $\mathcal{F}$ on Y, cap product with $[Y]$ induces a homotopy equivalence $C^*(Y; \mathcal{F}) \to C_*(Y; \mathcal{F} \wedge f^*\zeta_X)$.

(*b*) Let $[X]$ denote the composite map $S \xrightarrow{[Y]} C_*(Y; f^*\zeta_X) \to C_*(X; \zeta_X)$. For every local system of spectra $\mathcal{F}$ on X, cap product with $[X]$ induces a homotopy equivalence $C^*(X; \mathcal{F}) \to C_*(X; \mathcal{F} \wedge \zeta_X)$.

Remark 3. Condition (*a*) of Definition 2 guarantees that Y is a Poincare space with Spivak bundle $f^*\zeta_X$, and condition (*b*) guarantees that X is a Poincare space with Spivak bundle X.

A degree one structure on f can be regarded as a compatibility between the Poincare duality isomorphisms of X and Y. It can be thought of as consisting of two pieces of data:

(*i*) An identification of the Spivak bundle of Y with the pullback of the Spivak bundle of X.

(*ii*) An identification of the fundamental class of X with the pushforward of the fundamental class of Y (which makes sense by virtue of the datum (*i*)).

Remark 4. Let $f : Y \to X$ be a homotopy equivalence of Poincare spaces. Then f admits an essentially unique degree one structure. In other words, if X is a Poincare space, then the data of the pair $(\zeta_X, [X] : S \to C_*(X; \zeta_X))$ is determined uniquely up to a contractible space of choices.

Let $f : Y \to X$ be a map with a degree one structure $(\zeta_X, [Y])$. Then X and Y have visible symmetric signatures $\sigma_X^{vs} \in \Omega^\infty \mathbb{L}^{vs}(X, \zeta_X, R)$ and $\sigma_Y^{vs} \in \Omega^\infty \mathbb{L}^{vs}(Y, f^*\zeta_X, R)$. Our goal in this lecture is to prove the following result:

Theorem 5. *In the above situation, we can canonically construct a point* $\sigma_f^{vq} \in \Omega^\infty \mathbb{L}^{vq}(X, \zeta_X, R)$ *and a path from* $\sigma_X^{vs} + U(\sigma_f^{vq})$ *to* $f_! \sigma_Y^{vs}$ *in the space* $\Omega^\infty \mathbb{L}^{vs}(X, \zeta_X, R)$. *Here* U *denotes the canonical map of spectra* $\mathbb{L}^{vq}(X, \zeta_X, R) \to \mathbb{L}^{vs}(X, \zeta_X, R)$.

In fact, we can be even more precise. The visible symmetric signatures σ_X^{vs} and σ_Y^{vs} have canonical representatives $(\underline{R}, q_X)$ and $(\underline{R}, q_Y)$ by quadratic objects of $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$ and $\mathrm{Shv}_{\mathrm{lc}}(Y; \mathrm{RMod}_R)$, respectively. We will construct a canonical representative of σ_f^{vq}, so that the identity

$$\sigma_X^{vs} + U(\sigma_f^{vq}) \simeq f_! \sigma_Y^{vs}$$

is visible at the level of Poincare objects (that is, it comes from an equivalence of Poincare objects, rather than a bordism of Poincare objects).

In what follows, there is no loss of generality in treating the universal case where R is the sphere spectrum (equipped with the trivial involution). We will henceforth assume that we are in this case, and we will therefore omit mention of R in our notation.

Recall that $\mathbb{L}^{vs}(X, \zeta_X)$ can be identified with the L-theory of the finitely presented part of the ∞-category $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{Sp})$, equipped with the quadratic functor $Q^s_{\zeta_X}$ given by the formula

$$Q^s_{\zeta_X}(\mathcal{F}) = C_*(X; \zeta_X \wedge (\mathcal{F} \wedge \mathcal{F})^{h\Sigma_2}).$$

Similarly, $\mathbb{L}^{vq}(X, \zeta_X)$ is given by the L-theory of the same ∞-category, equipped with the quadratic functor

$$Q^q_{\zeta_X}(\mathcal{F}) = C_*(X; \zeta_X \wedge (\mathcal{F} \wedge \mathcal{F})_{h\Sigma_2}).$$

Using our assumption that X is a Poincare space with Spivak bundle ζ_X, we can rewrite

$$Q^s_{\zeta_X}(\mathcal{F}) = C^*(X; (\mathcal{F} \wedge \mathcal{F})^{h\Sigma_2}) \qquad Q^q_{\zeta_X}(\mathcal{F}) = C^*(X; (\mathcal{F} \wedge \mathcal{F})_{h\Sigma_2}).$$

In these terms, the visible symmetric signature σ_X^{vs} is easy to describe: it is represented by the Poincare object $(\underline{S}, q_X)$ where q_X classifies the evident global section of $(\underline{S} \wedge \underline{S})^{h\Sigma_2}$.

Let us now describe the pushforward of the visible symmetric signature of Y under the map f. This is given by a nondegenerate symmetric form q' on the local system $f_! \underline{S}$ on X. Let us describe this form more explicitly. We may assume without loss of generality that f is a fibration. For each point $x \in X$, we let Y_x

denote the fiber $f^{-1}\{x\}$. Unwinding the definitions, we see that the local system $f_! \underline{S}$ is given by the formula $x \mapsto \Sigma^\infty_+ Y_x$.

Let us now invoke our assumption that f is a degree one map. The fundamental class of $[Y]$ gives a map of spectra

$$S \to C_*(Y; f^*\zeta_X) \simeq C_*(X; f_! f^* \zeta_X) \simeq C_*(X; \zeta_X \wedge f_! \underline{S}) \simeq C^*(X; f_! \underline{S}).$$

We may view this as a map of local systems $u : \underline{S} \to f_! \underline{S}$. That is, for every point $x \in X$, we have a canonical map of spectra $u_x : S \to \Sigma^\infty_+ Y_x$. The condition that f be of degree one guarantees that the composite map

$$\underline{S} \to f_! \underline{S} \to \underline{S}$$

is homotopic to the identity. That is, each u_x can be regarded as a *section* of the canonical map $\Sigma^\infty_+ Y_x \to \Sigma^\infty_+ \{x\} \simeq S$.

Unwinding the definitions, we see that $q' \in \Omega^\infty Q^s_{\zeta_X}(f_! \underline{S})$ can be identified with the global section of $(f_! \underline{S} \wedge f_! \underline{S})^{h\Sigma_2}$ given by the composition

$$\underline{S} \xrightarrow{u} f_! \underline{S} \xrightarrow{\delta} (f_! \underline{S} \wedge f_! \underline{S})^{h\Sigma_2},$$

where δ is induced by the diagonal map $Y \to Y \times_X Y$. Fiberwise, this is given by the map of spectra

$$S \xrightarrow{u_x} \Sigma^\infty_+ Y_x \to (\Sigma^\infty_+ Y_x \wedge \Sigma^\infty_+ Y_x)^{h\Sigma_2},$$

where the second map is induced by the diagonal $Y_x \to Y_x \times Y_x$.

Theorem 5 can be obtained by fiberwise application of the following claim:

Proposition 6. *Let Z be a space (in our case of interest, the homotopy fiber of a map of Poincare spaces $f : X \to Y$), and suppose we are given a map of spectra $e : S \to \Sigma^\infty_+ Z$ which splits the canonical map $\Sigma^\infty_+ Z \to \Sigma^\infty_+ \{*\} \simeq S$. Let q denote the composition*

$$S \to \Sigma^\infty_+ Z \to (\Sigma^\infty_+ Z \wedge \Sigma^\infty_+ Z)^{h\Sigma_2}$$

where the second map is induced by the diagonal embedding $Z \to Z \times Z$. Then the quadratic object $(\Sigma^\infty_+ Z, q)$ splits (canonically!) as a direct sum $(S, q_+) \oplus (E, q_-)$, where $q_+ : S \to (S \wedge S)^{h\Sigma_2}$ is the evident map and $q_- : S \to (E \wedge E)^{h\Sigma_2}$ factors (canonically!) through the transfer map $(E \wedge E)_{h\Sigma_2} \to (E \wedge E)^{h\Sigma_2}$

The splitting $\Sigma^\infty_+ Z \simeq S \oplus E$ at the level of spectra is easy: it is determined by our choice of e. We therefore have an identification

$$(\Sigma^\infty_+ Z \wedge \Sigma^\infty_+ Z)^{h\Sigma_2} \simeq (S \wedge S)^{h\Sigma_2} \oplus (E \wedge S \oplus S \wedge E)^{h\Sigma_2} \oplus (E \wedge E)^{h\Sigma_2}.$$

We may therefore identify q with a trio of maps

$$q_+ : S \to (S \wedge S)^{h\Sigma_2}$$

$$q_0 : S \to (E \wedge S \oplus S \wedge E)^{h\Sigma_2} \simeq E$$

$$q_- : S \to (E \wedge E)^{h\Sigma_2}.$$

Unwinding the definitions, we see that q_0 is given by the composition

$$S \xrightarrow{e} \Sigma^\infty_+ Z \to E,$$

where the second map is projection onto the summand E. It follows that q_0 is canonically nullhomotopic, so that we obtain a direct sum decomposition of $(\Sigma^\infty_+ Z, q)$ as a quadratic object. Moreover, it is obvious that $q_+ : S \to (S \wedge S)^{h\Sigma_2}$ is as described. The only nontrivial point is to show that $q_- : S \to (E \wedge E)^{h\Sigma_2}$ admits a quadratic refinement (that is, it factors through the transfer map). We will take this up in the next lecture.