Transversality (Lecture 28)

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Let X be a Poincare space. Our goal is to determine whether or not there exists a homotopy equivalence $f : M \to X$, where M is a compact PL manifold. Suppose that this is the case. In this lecture, we will consider the first obstruction: X must admit a PL tangent bundle. More precisely, if we let $\chi : X \to \mathrm{Pic}(S)$ be the map classifying the Spivak normal bundle of X, then χ must factor (up to homotopy) through the classifying space $\mathbf{Z} \times \mathrm{BPL}$.

Let us be more precise. Let X be a finite complex, and choose a map $X \to \mathbf{Z} \times \mathrm{BPL}$, classifying a (stable) PL bundle on X which we will denote by ζ. Let ζ_0 denote the underlying spherical fibration of ζ (so that ζ_0 is a local system of invertible spectra on X). By a *normal structure* on ζ, we mean a map of spectra $S \to C_*(X; \zeta_0)$ which induces a homotopy equivalence $C^*(X; \mathcal{F}) \to C_*(X; \zeta_0 \wedge \mathcal{F})$ for every local system of spectra $\mathcal{F}$ on X. If ζ admits a normal structure, then X is a Poincare space, and a choice of normal structure amounts to a choice of homotopy equivalence $\zeta_0 \simeq \zeta_X$, where ζ_X denotes the Spivak normal fibration of X.

Let $\mathbb{S}^{tn}(X)$ denote a classifying space for pairs $(\zeta, [X])$, where ζ is a stable PL bundle on X and $[X] : S \to C_*(X; \zeta_0)$ is a normal structure on ζ. We will refer to $\mathbb{S}^{tn}(X)$ as the space of *tangential normal structures* on X. If X is a Poincare space, then $\mathbb{S}^{tn}(X)$ can be described as the homotopy fiber of the map

$$(\mathbf{Z} \times \mathrm{BPL})^X \to \mathrm{Pic}(S)^X,$$

taken over the map $X \to \mathrm{Pic}(S)$ classifying the Spivak normal fibration of X. (If X is not a Poincare space, then $\mathbb{S}^{tn}(X)$ is empty.)

Our goal in this lecture is to obtain a more geometric description of the space $\mathbb{S}^{tn}(X)$.

Definition 1. Let X be a finite complex and let ζ be a stable PL bundle over X. A *normal map* to X consists of the following data:

(*a*) A map $f : M \to X$, where M is a compact PL manifold.

(*b*) An equivalence $\alpha : f^*\zeta \simeq -T_M$ of stable PL bundles (where T_M denotes the PL tangent bundle of M).

In this case, we obtain a map of spectra

$$S \stackrel{[M]}{\to} C_*(M; \zeta_M) \stackrel{\alpha}{\simeq} C_*(M; f^*\zeta_0) \to C_*(X; \zeta_0).$$

where ζ_M denotes the Spivak normal bundle of M (which underlies the stable PL bundle $-T_M$) and $[M]$ denotes the fundamental class of M. We will say that f is a *degree one normal map* if this composite map determines a normal structure on ζ.

We will generally abuse notation by identifying a normal map $(f, \alpha) : (M, -T_M) \to (X, \zeta)$ with the underlying map $f : M \to X$. Note that if there exists a degree one normal map $M \to X$, then X must be a Poincare space. Conversely, suppose that X is a Poincare space and that ζ is a stable PL bundle refining the Spivak normal bundle. Then we have $C_*(X; \zeta_0) \simeq C^*(X; \underline{S})$, so that any normal map $f : M \to X$ determines a section of the trivial bundle $\underline{S}$. The condition that f be degree one is the condition that this section determines an equivalence from $\underline{S}$ to itself. More concretely, it means that the fundamental class $[M]$ pushes forward to a fundamental class of X.

Suppose we are given a finite polyhedron X and a stable PL bundle ζ on X. If there exists a degree one normal map $f : M \to X$, then ζ admits a normal structure. The converse is true as well. Suppose we are given a normal structure $u : S \to C_*(X;\zeta_0)$. We can identify $C_*(X;\zeta_0)$ with the Thom spectrum X^{ζ_0}. This Thom spectrum can be described more concretely as follows: choose a PL disk bundle $D \to X$ and an equivalence of stable PL bundles $D \simeq \zeta \oplus \underline{\mathbb{R}}^n$, for $n \gg 0$. Then $\Sigma^n X^{\zeta_0}$ is the suspension spectrum of the Thom space $D/\partial D$ of the disk bundle D. In particular, after enlarging n, we can assume that u is induced by a map of pointed spaces $e : S^n \to D/\partial D$. Choose a PL section of the map $D \to X$ whose image Z is disjoint from ∂D (which we will refer to as the *zero section*). Modifying e by a homotopy, we may assume that it is piecewise linear and in general position with respect to the zero section (if we work in the setting of vector bundles rather than PL bundles, we should assume here that e is transverse to the zero section). Then $M = e^{-1}Z$ is a compact PL submanifold of S^n. Moreover, we have an equivalence of stable PL bundles $T_M \oplus e^* D \simeq \underline{R}^n$, so that e induces a normal map $M \to X$. By construction, this normal map carries the Thom-Pontryagin collapse map $S \to M^{-T_M}$ to the normal structure u, and is therefore of degree one.

The discussion above involved some arbitrary choices, so the normal map $f : M \to X$ is not uniquely determined by the normal structure on ζ. Let us now explain how to account for this ambiguity.

Definition 2. Let $k \geq 0$ be an integer, let X be a polyhedron, and let ζ be a stable PL bundle on X. A Δ^k*-family of normal maps to* X consists of the following data:

- (a) A map $M \to X \times \Delta^k$, inducing a neat map $e : M \to \Delta^k$ and a map $f : M \to X$, where M is a PL manifold with boundary $\partial M = e^{-1}\partial \Delta^k$.
- (b) An equivalence of stable PL bundles $\alpha : T_M \simeq e^* T_{\Delta^k} - f^*\zeta$.

In this case, for every vertex v of Δ^k, f induces a normal map from the PL manifold $M_v = g^{-1}\{v\}$ to X. We will say that $(M \to \Delta^k \times X, \alpha)$ has *degree one* if the induced normal map $M_v \to X$ has degree one. Note that this condition is independent of the choice of v (because the underlying map $S \to C_*(X;\zeta_0)$ is independent of v, up to homotopy).

Definition 3. Let $k \geq 0$ be an integer and let X be a finite complex. We let $\mathbb{S}^n(X)_k$ denote a classifying space for the following data:

- (i) A stable PL bundle ζ on X.
- (ii) A Δ^k-family of normal maps $M \to \Delta^k \times X$ having degree one.

The definition of $\mathbb{S}^n(X)_k$ is functorial in k; we therefore obtain a simplicial space $\mathbb{S}^n(X)_\bullet$. We will denote the realization of $\mathbb{S}^n(X)_\bullet$ by $\mathbb{S}^n(X)$, and refer to it as the *normal structure space of* X.

Every Δ^k-family of normal maps to a pair (X,ζ) determines a normal structure on the stable PL bundle ζ (given by the pushforward of the fundamental class of M along the map $M \to X \times \Delta^k$). We therefore obtain a map

$$\mathbb{S}^n(X) \to \mathbb{S}^{tn}(X).$$

Proposition 4. *The map $\mathbb{S}^n(X) \to \mathbb{S}^{tn}(X)$ is a homotopy equivalence.*

In other words, any stable PL structure on the Spivak normal bundle of a Poincare complex X determines a normal map $f : M \to X$, where M is well-defined up to (normal) bordism.

Proof. Let $Z_\bullet$ denote the constant simplicial space taking the value $\mathbb{S}^{tn}(X)$. It will suffice to show that the map of simplicial spaces $\mathbb{S}^n(X)_\bullet \to Z_\bullet$ is a trivial Kan fibration. In other words, for every integer k, we must show that the map

$$\mathbb{S}^n(X)_k \to M_k(\mathbb{S}^n(X)) \times_{\mathbb{S}^{tn}(X)^{\partial \Delta^k}} \mathbb{S}^{tn}(X)$$

is surjective on connected components. When $k = 0$, this amounts to the argument given above. The proof in general is similar (it involves general position/ transversality arguments). $\square$

Now let X be an arbitrary Poincare space, and suppose we are given a homotopy equivalence $f : M \to X$, where M is a compact PL manifold. Since f is a homotopy equivalence, we can choose a stable PL bundle ζ on X and an equivalence $-T_M \simeq f^* \zeta$, so that f has the structure of a degree one normal map. More generally, if we are given a datum

$$(f, e, j) : M \hookrightarrow X \times \Delta^k \times \mathbb{R}^m$$

defining a k-simplex of the structure space $\mathbb{S}(X)$, then there is an essentially unique PL bundle ζ on X such that $f^* \zeta \simeq e^* T_{\Delta^k} - T_M$, for which the pair (f, e) becomes a Δ^k-family of normal maps determining a point of $\mathbb{S}^n(X)_k$. We therefore obtain a map of structure spaces

$$\mathbb{S}(X) \to \mathbb{S}^n(X) \simeq \mathbb{S}^{tn}(X) \simeq \mathrm{fib}((\mathbf{Z} \times \mathrm{BPL})^X \to \mathrm{Pic}(S)^X).$$

Our goal for the rest of this course will be to analyze the homotopy fiber of *this* map. The work of this lecture gives us a good starting point: a point of $\mathbb{S}^n(X)$ determines a degree one normal map $f : M \to X$, where M is well-defined up to (normal) bordism. We now wish to determine if it is possible to modify M by a (normal) bordism so as to replace f by a homotopy equivalence.

Remark 5. Let $(X, \partial X)$ be a Poincare pair, where ∂X is a PL manifold. We can then define relative versions of the structure spaces $\mathbb{S}^n(X)$ and $\mathbb{S}^{tn}(X)$, and the above discussion carries over essentially without change.