The Structure Space (Lecture 27)

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Let X be a Poincare space. The goal of this course is to answer the following:

Question 1. When is X homotopy equivalent to a manifold?

In this lecture, we will formulate this question more precisely by introducing the (piecewise-linear) *structure space* of X, which we will denote by $\mathbb{S}(X)$. We first need to introduce a bit of terminology.

Let $f : M \to \Delta^k$ be a map of finite polyhedra. We will say that f is *neat* if, for every face $\tau \subseteq \Delta^k$ of codimension p, the following condition is satisfied:

($*$) There exists a neighborhood U of τ which is PL homeomorphic to $\tau \times \mathbb{R}^p_{\geq 0}$, such that the induced map $f^{-1}(U) \to \tau \times \mathbb{R}^p_{\geq 0} \to \mathbb{R}^p_{\geq 0}$ is a (necessarily trivial) PL fiber bundle.

Example 2. If f is a fiber bundle, then it is neat.

We will be particularly interested in the case where M is a PL manifold with boundary of dimension n. In this case, if $f : M \to \Delta^k$ is neat, then for every face $\tau \subseteq \Delta^k$ of codimension p, the inverse image $f^{-1}\tau$ is a PL manifold with boundary of dimension $n - p$. Moreover, the boundary of $f^{-1}\tau$ is the intersection $f^{-1}\tau \cap \partial M$.

Example 3. In the special case $k = 1$, a neat map $f : M \to \Delta^k$ determines a bordism from the PL manifold $f^{-1}\{0\}$ to the PL manifold $f^{-1}\{1\}$. Conversely, any bordism M between PL manifolds M_0 and M_1 admits neat map $M \to \Delta^1$, choosing any PL function $M \to [0, 1]$ which is sufficiently well-behaved near the two ends.

Now suppose that $(X, \partial X)$ is a Poincare pair. We may assume without loss of generality that X is a finite polyhedron and that ∂X is given a closed subpolyhedron of X. Let us suppose furthermore that ∂X is a PL manifold. For every integer m, we define a simplicial set $\mathbb{S}(X, \mathbb{R}^m)$ as follows: a k-simplex of $\mathbb{S}(X, \mathbb{R}^m)$ is a finite subpolyhedron $M \subseteq X \times \mathbb{R}^m \times \Delta^k$ satisfying the following conditions:

(a) M is a compact PL manifold with boundary.

(b) Let $N \subseteq M$ be the closure of boundary of $M \cap X \times \mathbb{R}^m \times (\Delta^k - \partial \Delta^k)$. Then the induced map $F : M \to X \times \Delta^k$ induces a PL homeomorphism $N \to \partial X \times \Delta^k$.

(c) Let $f : M \to \Delta^k$ be the composition of F with projection onto the second factor. Then f is neat.

(d) For every face $\tau \subseteq \Delta^k$, the induced map $f^{-1}(\tau) \to X$ is a homotopy equivalence.

We let $\mathbb{S}(X)$ denote the direct limit $\varinjlim_m \mathbb{S}(X, \mathbb{R}^m)$. We will refer to $\mathbb{S}(X)$ as the *structure space* of the pair $(X, \partial X)$.

Remark 4. Assume for simplicity that the boundary ∂M is empty. Informally, giving a k-dimensional simplex of $\mathbb{S}(X)$ is equivalent to giving a neat map $f : M \to \Delta^k$ such that, for each simplex $\tau \subseteq \Delta^k$, the inverse image $f^{-1}\tau$ is equipped with a homotopy equivalence to X (which is compatible with enlargement of τ). There is also an auxiliary datum of a map from M into $\mathbb{R}^m$ for $m \gg 0$, but this data should be ignored (it is there to rigidify the problem, so that the k-simplices of $\mathbb{S}(X)$ form a set rather than a topological groupoid).

Remark 5. The structure space $\mathbb{S}(X)$ defined above is a Kan complex.

Following Remark 4 (and still assuming $\partial X = \emptyset$), we can describe the low-dimensional simplices of $\mathbb{S}(X)$ as follows:

- Giving a 0-simplex of $\mathbb{S}(X)$ is equivalent to giving a compact PL manifold M equipped with a homotopy equivalence $M \to X$.
- A 1-simplex of $\mathbb{S}(X)$ joining 0-simplex corresponding to $f_0 : M \to X$ and $f_1 : N \to X$ is a bordism B from M to N, together with a map $f : B \to X$ extending f_0 and f_1, such that f is a homotopy equivalence. It follows that the inclusions $M \hookrightarrow B \hookleftarrow N$ are homotopy equivalences: that is, B is an *h-cobordism* from M to N.
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In particular, $\pi_0 \mathbb{S}(X)$ is the collection of equivalence classes of PL manifolds equipped with a homotopy equivalence to X, where the equivalence relation is given by h-cobordism.

Variant 6. Suppose that we modify the definition of $\mathbb{S}(X)$ by adding the following additional condition on a k-simplex of $\mathbb{S}(X; \mathbb{R}^m)$:

(e) The map $f : M \to \Delta^k$ is a PL fiber bundle.

We then obtain a different notion of structure space $\mathbb{S}^+(X)$, where $\pi_0 \mathbb{S}^+(X)$ is the collection of PL manifolds equipped with a homotopy equivalence to X, up to the equivalence relation given by PL homeomorphism. We have a map of structure spaces $\mathbb{S}^+(X) \to \mathbb{S}(X)$. The structure space $\mathbb{S}^+(X)$ is much more difficult to describe than $\mathbb{S}(X)$: to obtain information about it, one must go far beyond the tools we have introduced in this class.

We can now restate our main goal as follows:

Question 7. Given a Poincare space X (or a Poincare pair $(X, \partial X)$ where ∂X is a PL manifold), what is the homotopy type of the structure space $\mathbb{S}(X)$?

To answer this question, let us fix an A_∞-ring R with involution. Assume for simplicity that $\partial X = \emptyset$. Let M be a compact closed PL manifold, and let ζ_X and ζ_M be the Spivak normal bundles to X and M, respectively. Since M is a PL manifold, the constant sheaf $\underline{R}$ is a Poincare object of $(\mathrm{Shv}_{\mathrm{const}}(M; \mathrm{LMod}_R^{\mathrm{fp}}), Q^s_{\zeta_M})$. Suppose that we are given a homotopy equivalence $f : M \to X$, so that $\zeta_M \simeq f^* \zeta_X$. Then $f_* \underline{R}$ determines a Poincare object of $(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}), Q^s_{\zeta_X})$, and therefore a point of the space

$$\mathrm{Poinc}(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}), Q^s_{\zeta_X}).$$

Elaborating on this construction, we obtain a map of simplicial spaces

$$\mathbb{S}(X)_\bullet \to \mathrm{Poinc}(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}), Q^s_{\zeta_X})_\bullet.$$

Passing to geometric realizations, we get a map

$$\mathbb{S}(X) \to L(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}), Q^s_{\zeta_X}) \simeq \Omega^\infty \mathbb{L}^s(X, \zeta_X, R).$$

Let us denote this map by σ^s (for "symmetric signature"). It is closely related to the visible symmetric signature of X defined in the previous lecture. More precisely, the composite map

$$\mathbb{S}(X) \xrightarrow{\sigma} \Omega^\infty \mathbb{L}^s(X, \zeta_X, R) \to \Omega^\infty \mathbb{L}^{vs}(X, \zeta_X, R)$$

is homotopic to a constant map, taking the value σ_X^{vs}. (Unwinding the definitions, this amounts to the observation that when $f : M \to X$ is a homotopy equivalence, then the functor $f_!$ carries the locally constant sheaf $\underline{R}$ on M to the locally constant sheaf $\underline{R}$ on X.)

We can summarize the situation informally as follows. If X is a Poincare space, we can define a visible symmetric signature $\sigma_X^{vs} \in \Omega^\infty \mathbb{L}^{vs}(X, \zeta_X, R)$. This invariant reflects the fact that X satisfies a global form of Poincare duality (that is, we have a Poincare duality isomorphism for locally constant sheaves on X). However, if we are given a homotopy equivalence $f : M \to X$ where M is a PL manifold, then we can say more. The manifold M satisfies Poincare duality not only globally but also locally: that is, the constant sheaf on M is self-dual in the category of all (constructible) sheaves on M, rather than merely the locally constant sheaves. We therefore obtain an invariant $\sigma^s(M) \in \Omega^\infty(\mathbb{L}^s(X, \zeta_X, R)$, which refines the visible symmetric signature (and depends on the homotopy equivalence $f : M \to X$).

To describe the situation more systematically, let F_R denote the homotopy fiber of the map $\Omega^\infty \mathbf{L}^s(X, \zeta_X, R) \to \Omega^\infty \mathbf{L}^{vs}(X, \zeta_X, R)$ (taken over the point σ_X^{vs}). The above constructions determine a map of spaces $\mathbb{S}(X) \to F$.

Non-Theorem 1. *Let X be a Poincare space and let $R = \mathbf{Z}$. Then the map $\mathbb{S}(X) \to F_R$ is a homotopy equivalence.*

Assertion 1 is not quite right. To correct it, we need to make three modifications:

(1) We must work in the setting of topological manifolds, rather than piecewise linear manifolds. (In dimensions ≥ 5, there is not too much difference between the two. As a consequence, Assertion 1 is not too far from being correct in the PL setting, as we will see later.)

(2) We must restrict our attention to the case where the Poincare space X has dimension at least 5 (the domain of high-dimensional topology). If we work in the topological setting, this can be slightly relaxed: one can allow 4-dimensional Poincare spaces provided that their fundamental groups are not too big.

(3) We need to replace F by an appropriate subset $F_R^0 \subseteq F_R$ which we now describe. Recall that $\mathbf{L}^s(X, \zeta_X, R)$ can be identified with $C_*(X; \mathbf{L}(\mathrm{LMod}_R^{\mathrm{fp}}, \zeta_X(x) \wedge Q^s))$. Since X is a Poincare space, we can rewrite this spectrum as

$$C^*(X; \zeta_X^{-1} \mathbf{L}(\mathrm{LMod}_R^{\mathrm{fp}}, \zeta_X(x) \wedge Q^s).$$

That is, we can identify points of $\Omega^\infty \mathbb{L}^s(X, \zeta_X, R)$ with maps of local systems $\zeta_X \to \mathcal{F}$, where $\mathcal{F}$ is the local system of spectra which assigns to each point $x \in X$ the spectrum $\mathbf{L}(\mathrm{LMod}_R^{\mathrm{fp}}, \zeta_X(x) \wedge Q^s)$. We have $R = \mathbf{Z}$, so that $\mathbb{L}^s(R)$ is a ring spectrum. Consequently, any point of $\Omega^\infty \mathbb{L}^s(X, \zeta_X, R)$ determines, for each $x \in X$, a map of spectra

$$\theta_x : \zeta_X(x) \wedge \mathbb{L}^s(R) \to \mathbb{L}(\mathrm{LMod}_R^{\mathrm{fp}}, \zeta_X(x) \wedge Q^s).$$

Let $(\Omega^\infty \mathbb{L}^s(X, \zeta_X, R))^\times$ denote the subspace of of $\Omega^\infty \mathbb{L}^s(X, \zeta_X, R)$ corresponding to those points for which each θ_x is a homotopy equivalence, and let F_R^0 denote the homotopy fiber of the induced map

$$\Omega^\infty \mathbb{L}^s(X, \zeta_X, R))^\times \to \Omega^\infty \mathbb{L}^{vs}(X, \zeta_X, R)$$

(taken over the visible symmetric signature of R). Note that the map $\mathbb{S}(X) \to F_R$ factors through F_R^0: when X is a PL manifold, the map θ_x defined above is determined by the orientation of $\mathbb{L}^s(\mathbf{Z})$ with respect to the PL tangent bundle of X.

With these amendments, assertion 1 becomes a theorem. This is Ranicki's theory of the *total surgery obstruction*. We will not prove the theorem in this course (though we will get close), primarily because we do not want to talk about topological manifolds.

We close this lecture by observing that the above analysis can be generalized to the case where $(X, \partial X)$ is a Poincare pair with ∂X a piecewise linear manifold. In this case, we have defined a visible symmetric signature

$$\sigma_{(X, \partial X)}^{vs} \in \Omega^\infty \mathbb{L}^{vs}(X, \partial X, \zeta_X, R) = \Omega^\infty \operatorname{cofib}(\mathbb{L}^{vs}(\partial X, \zeta_X | \partial X, R) \to \mathbb{L}^{vs}(X, \zeta_X, R)).$$

Let $\mathbb{L}^s(X, \partial X, \zeta_X, R)$ denote the cofiber of the map

$$\mathbb{L}^s(\partial X, \zeta_X | \partial X, R) \to \mathbb{L}^s(X, \zeta_X, R),$$

and let F_R denote the homotopy fiber of the map

$$\Omega^\infty \mathbb{L}^s(X, \partial X, \zeta_X, R) \to \Omega^\infty \mathbb{L}^{vs}(X, \partial X, \zeta_X, R)$$

over the point $\sigma^{vs}_{(X, \partial X)}$. The constructions above generalize to give a map $\mathbb{S}(X) \to F_R$. Roughly speaking, our goal over the next few lectures is to show that this map is close to being a homotopy equivalence.