The Hirzebruch Signature Formula (Lecture 25)

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In the last lecture, we defined a characteristic class $l(\zeta) \in \prod_k \mathrm{H}^{4k}(X; \mathbf{Q})$ associated to an oriented PL bundle on a space X, given by

$$l(\zeta) = l_0(\zeta) + l_4(\zeta) + \cdots$$

If M is a compact oriented PL manifold of dimension $4k$ and ζ is its (stable) normal bundle, then the signature of M is given by

$$\sigma_M = l_{4k}(\zeta)[M].$$

To obtain the Hirzebruch signature formula, we need to describe the characteristic class $l(\zeta)$ explicitly. For simplicity, let us restrict our attention to the case where ζ comes from a (virtual) vector bundle of rank zero, so that ζ is classified by a map $X \xrightarrow{\chi} \mathrm{BSO} = \varinjlim \mathrm{BSO}(n)$. Then ζ is the pullback of a universal bundle ζ_0 on BSO, so that $l(\zeta) = \chi^* l(\zeta_0)$. It will therefore suffice to describe $l(\zeta_0)$ as an element of the cohomology ring $\prod_k \mathrm{H}^{4k}(\mathrm{BSO}; \mathbf{Q}) \simeq \mathbf{Q}[[p_1, p_2, \ldots]]$; here p_i denote the universal Pontryagin classes.

Remark 1. Our restriction to virtual vector bundles of rank zero involves no essential loss of generality. First, the characteristic class $l(\zeta)$ does not change if we add a trivial bundle to ζ, so we may as well assume that ζ is of degree zero and classified by a map $X \to \mathrm{BSPL}$ (where SPL denotes the index two subgroup of PL corresponding to homeomorphisms which preserve orientation). At the very end of this course, we will see that the map $\mathrm{BSO} \to \mathrm{BSPL}$ is a rational homotopy equivalence.

Every complex vector bundle can be regarded as an oriented real vector bundle. This observation determines a map of classifying spaces $\mathrm{BU} \to \mathrm{BSO}$. The induced map

$$\mathrm{H}^*(\mathrm{BSO}; \mathbf{Q}) \to \mathrm{H}^*(\mathrm{BU}; \mathbf{Q})$$

is injective on cohomology. Consequently, to describe $l(\zeta_0)$, it will suffice to describe its image in $\mathrm{H}^*(\mathrm{BU}(n); \mathbf{Q})$ for every integer n. According to the splitting principle for complex line vector bundles, the canonical map

$$\mathrm{H}^*(\mathrm{BU}(n); \mathbf{Q}) \to \mathrm{H}^*(\mathrm{BU}(1)^n; \mathbf{Q}) \simeq \mathrm{H}^*(\mathrm{BU}(1); \mathbf{Q})^{\otimes n}$$

is injective on cohomology. In other words, it suffices to describe $l(\zeta)$ in the special case where ζ comes from a complex vector bundle of rank n that is written as a direct sum $\zeta_1 \oplus \cdots \oplus \zeta_n$ of complex line bundles. Our construction of L-theory orientations is compatible with the formation of direct sums of PL disk bundles, so we have

$$l(\zeta) = \prod_{1 \leq i \leq n} l(\zeta_i).$$

We are therefore reduced to describing $l(\zeta)$ when ζ arises from a complex line bundle. Here it suffices to treat the case $X = \mathrm{BU}(1) = \mathbf{C}\mathrm{P}^\infty$, so that $\mathrm{H}^*(X; \mathbf{Q}) \simeq \mathbf{Q}[[H]]$ where $H \in \mathrm{H}^2(X; \mathbf{Q})$ denotes the first Chern class of the universal line bundle. We can therefore write $l(\zeta) = f(H)$ for some power series f with rational coefficients, which are yet to be determined. Note that since $f(H)$ lies in $\prod_k \mathrm{H}^{4k}(X; \mathbf{Q})$, the power series f is even: that is, we have $f(H) = f(-H)$, so that f can be written as a function of H^2.

We can determine the power series f by applying the signature formula to the manifolds $\mathbf{C}\mathrm{P}^n$, where n ranges over the integers. Note that the tangent bundle T to $\mathbf{C}\mathrm{P}^n$ fits into an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to T \to 0.$$

If we let $\zeta_{\mathbf{C}\mathrm{P}^n}$ denote the normal bundle to $\mathbf{C}\mathrm{P}^n$, we have $l(\zeta_{\mathbf{C}\mathrm{P}^n}) = l(T)^{-1} = l(\mathcal{O}(1))^{-n-1} = f(H)^{-n-1}$. The fundamental class of $\mathbf{C}\mathrm{P}^n$ is dual to H^n. It follows that the signature $\sigma(\mathbf{C}\mathrm{P}^n)$ of $\mathbf{C}\mathrm{P}^n$ is given by the coefficient of H^n in the power series $f(H)^{-n-1}$. Equivalently, the signature of $\mathbf{C}\mathrm{P}^n$ is given by the residue of (formal) differential $(\frac{1}{Hf(H)})^{n+1}dH$.

We can describe this residue more explicitly using the Lagrange inversion formula. Since the constant term of f is invertible, $Hf(H)$ is an invertible power series. That is, if we set $U = Hf(H)$, then there is a power series g such $H = g(U) = \sum c_i U^i$. Then $(\frac{1}{Hf(H)})^{n+1}dH = \frac{1}{U^{n+1}}dg(U) = \frac{1}{U^{n+1}}g'(U)dU$. The residue of this formal series is given by $(n+1)c_{n+1}$. It follows that

$$g(U) == \sum \frac{\sigma(\mathbf{C}\mathrm{P}^n)}{n+1} U^{n+1} = U + \frac{U^3}{3} + \frac{U^5}{5} + \cdots .$$

This power series describes the transcendental function $\tanh^{-1}$, so that $Hf(H) = \tanh(H)$ and $f(H) = \frac{\tanh(H)}{H}$.

Let us now make the above formula more explicit. When ζ comes from a complex line bundle, it has a first Chern class $c_1(\zeta)$, and $l(\zeta)$ is given by $f(c_1(\zeta))$. Since f is really a function of H^2, we can write $f(c_1(\zeta))$ as a power series in the Pontryagin class $p_1(\zeta) = c_1(\zeta)^2$. Informally, we have

$$l(\zeta) = \frac{\tanh(\sqrt{p_1(\zeta)})}{\sqrt{p_1(\zeta)}}$$

If ζ is a sum of complex line bundles $\zeta_1, \zeta_2, \ldots, \zeta_n$, then we have

$$l(\zeta) = \prod_i l(\zeta_i) = \prod_i \frac{\tanh(\sqrt{p_1(\zeta_i)})}{\sqrt(p_1(\zeta_i)}.$$

This can be written as an (infinite) sum of symmetric polynomials in the variables $p_1(\zeta_1), \ldots, p_1(\zeta_n)$. It can therefore be *rewritten* as an (infinite) sum of polynomials in the elementary symmetric polynomials in the variables $p_1(\zeta_1), \ldots, p_1(\zeta_n)$, which are the Pontryagin classes $p_1(\zeta), p_2(\zeta), \ldots, p_n(\zeta)$. Taking the limit as $n \to \infty$, we obtain a formula for $l(\zeta)$ in terms of the Pontryagin classes of ζ (which is valid for any orientable real vector bundle, by virtue of the above remarks concerning injectivity on cohomology).

We can recast the above discussion in the language of complex oriented cohomology theories. The symmetric L-theory spectrum $\mathbb{L}^s(\mathbf{Z})$ is orientable with respect to all oriented real vector bundles, and in particular has a canonical complex orientation. Following Quillen, we can associate to $\mathbb{L}^s(\mathbf{Z})$ a formal group law. The above calculation can be interpreted as saying that (modulo torsion) this formal group law is given by

$$(X, Y) \mapsto \tanh(\tanh^{-1}(X) + \tanh^{-1}(Y)).$$

That is, tanh is the inverse of the logarithm for the formal group law determined by our complex orientation on $\mathbb{L}^s$. Let

$$p(X) = X + \frac{X^2}{2} + \frac{X^3}{6} + \cdots = e^X - 1$$

be the inverse of the logarithm for the multiplicative formal group law. Then

$$\tanh(X) = \frac{e^X - e^{-X}}{e^X + e^{-X}} = \frac{e^{2X} - 1}{e^{2X} + 1} = \frac{p(2X)}{p(2X) + 2}.$$

After inverting 2, we see that $\tanh(X)$ and $p(X)$ differ by an invertible change of coordinate. It follows that the formal group law

$$(X, Y) \mapsto \tanh(\tanh^{-1}(X) + \tanh^{-1}(Y))$$

is isomorphic to the multiplicative formal group after inverting 2. Using Landweber exactness, we see that the spectrum $\mathbb{L}^s(\mathbf{Z})[\frac{1}{2}] \simeq \mathbb{L}^q(\mathbf{Z})[\frac{1}{2}]$ is determined up to homotopy equivalence by the graded ring $\pi_* \mathbb{L}^s(\mathbf{Z})[\frac{1}{2}] \simeq \mathbf{Z}[\frac{1}{2}][t^{\pm 1}]$ together with the associated formal group, which is the multiplicative group. This yields a proof of the following result:

Proposition 2. *There is a homotopy equivalence of spectra* $\mathbb{L}^s(\mathbf{Z})[\frac{1}{2}] \simeq \mathrm{KO}[\frac{1}{2}]$, *where* KO *denotes the real* K*-theory spectrum.*