L-Theory Orientations of Manifolds (Lecture 24)

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Let us begin by introducing some terminology to systematize the ideas of the previous lecture. Let X be a polyhedron. A *closed n-disk bundle* over X is a map of polyhedra $q : D \to X$ such that every point $x \in X$ has an open neighborhood U for which there is a PL homeomorphism $q^{-1}U \simeq U \times \Delta^n$ (which commutes with the projection to U).

There is a canonical bijection between isomorphism classes of closed n-disk bundles over X and homotopy classes of maps $X \to B\,\mathrm{Disk}(n)$, where $B\,\mathrm{Disk}(n)$ denotes the classifying space of the (simplicial) group $\mathrm{Disk}(n)$ of PL homeomorphisms of Δ^n.

The disjoint union $\coprod_n B\,\mathrm{Disk}(n)$ is equipped with a multiplication which is associative up to coherent homotopy, classifying the formation of products of closed n-disk bundles. We can describe the group completion of $\coprod_n B\,\mathrm{Disk}(n)$ as a product $\mathbf{Z} \times \mathrm{BPL}$, where BPL is the direct limit $\varinjlim_n B\,\mathrm{Disk}(n)$.

Let $\mathrm{Pic}(S)$ denote the classifying space for invertible spectra (so that homotopy classes of maps $X \to \mathrm{Pic}(S)$ correspond to equivalence classes of spherical fibrations over X). Every closed disk bundle $q : D \to X$ has an associated spherical fibration, given by $x \mapsto \Sigma^\infty(D_x / \partial\, D_x)$. This construction determines a map $\coprod_n B\,\mathrm{Disk}(n) \to \mathrm{Pic}(S)$, which is multiplicative up to coherent homotopy and therefore extends to a map $\mathbf{Z} \times \mathrm{BPL} \to \mathrm{Pic}(S)$.

Fix $(\mathcal{C}, Q)$ as above. Over the space $\mathrm{Pic}(S)$, we have two local systems of spectra: one given by the formula $E \mapsto \mathbb{L}(\mathcal{C}, E \wedge Q)$ and one given by the formula $E \mapsto E \wedge \mathbb{L}(\mathcal{C}, Q)$. The constructions of the previous lecture show that these spherical fibrations are (canonically) equivalent when restricted to $\mathbf{Z} \times \mathrm{BPL}$.:

Proposition 1. *Let X be a finite polyhedron with triangulation T, $\mathcal{C}$ a stable ∞-category equipped with a nondegenerate quadratic functor Q, and ζ a spherical fibration on X, classified by a map $X \to \mathrm{Pic}(S)$. Suppose that this classifying map factors through $\mathbf{Z} \times \mathrm{BPL}$ (that is, that the spherical fibration ζ arises from a closed disk bundle, at least stably). Then there is a homotopy equivalence (depending canonically on the factorization)*

$$\mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q_\zeta) \simeq \varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}(\mathcal{C}, Q).$$

Now suppose that M is a compact PL manifold with boundary and let ζ_M be its normal fibration. Then ζ_M factors canonically through $\mathbf{Z} \times \mathrm{BPL}$. Let T be a triangulation of M, so that we have a canonical orientation

$$[M] : S \to \varinjlim_{\tau \in T} \begin{cases} 0 & \text{if } \tau \subseteq \partial\, M \\ \zeta_M(\tau) & \text{otherwise.} \end{cases}$$

Smashing with $\mathbb{L}(\mathcal{C}, Q)$ and using Proposition 1, we obtain a map

$$\begin{aligned} \mathbb{L}(\mathcal{C}, Q) &\to \varinjlim_{\tau \in T} \begin{cases} 0 & \text{if } \tau \subseteq \partial\, M \\ \zeta_M(\tau) \wedge \mathbb{L}(\mathcal{C}, Q) \end{cases} \\ &\simeq \varinjlim_{\tau \in T} \begin{cases} 0 & \text{if } \tau \subseteq \partial\, M \\ \mathbb{L}(\mathcal{C}, \zeta_M(\tau) \wedge Q) \end{cases} \\ &\simeq \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(M, \partial\, M; \mathcal{C}), Q_{\zeta_M}). \end{aligned}$$

This can be identified with the map constructed in the previous lecture, which carries each Poincare object (C, q) to a Poincare object $(\underline{C}, q_{[M]})$, where $\underline{C}$ denotes the constant sheaf taking the value C.

We can also make the analysis of the preceding discussion read in a different way. Let us suppose that $\mathcal{C} = \mathcal{D}^{\mathrm{fp}}(\mathbf{Z})$ is the ∞-category of perfect complexes of $\mathbf{Z}$-modules, and let Q be either Q^q or Q^s. Then Q is a spectrum valued functor which factors through the ∞-category of $\mathbf{Z}$-module spectra. It follows that for every spectrum E, we can write $E \wedge Q \simeq (E \wedge \mathbf{Z}) \wedge_{\mathbf{Z}} Q$, so that $E \wedge Q$ depends only on the generalized Eilenberg-MacLane spectrum $E \wedge \mathbf{Z}$. Let ζ be a spherical fibration on a polyhedron X, and suppose that ζ assigns to each point $x \in X$ a spectrum $\zeta(x)$ which is homotopy equivalent to $\Sigma^n S$. Suppose further that ζ is orientable. A choice of orientation determines a canonical homotopy equivalence of each $\zeta(x) \wedge \mathbf{Z}$ with $\Sigma^n \mathbf{Z}$, and therefore a natural isomorphism $Q_\zeta \simeq \Sigma^n Q$. It follows that we obtain a canonical homotopy equivalence

$$\varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}(\mathcal{C}, Q) \simeq \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q_\zeta) \simeq \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), \Sigma^n Q) \simeq \Sigma^n \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q) \simeq \Sigma^n (X \wedge \mathbb{L}(\mathcal{C}, Q)).$$

This proves:

Proposition 2. *If ζ is an oriented spherical fibration (of dimension n) on X classified by a map $X \to \mathrm{Pic}(S)$ which factors through $\mathbf{Z} \times \mathrm{BPL}$, then we have homotopy equivalences (depending canonically on the choice of factorization)*

$$\varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}^q(\mathbf{Z}) \simeq \Sigma^n (X \wedge \mathbb{L}^q(\mathbf{Z}))$$

$$\varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}^s(\mathbf{Z}) \simeq \Sigma^n (X \wedge \mathbb{L}^s(\mathbf{Z}))$$

Let $\mathbb{L}$ denote the symmetric L-theory spectrum $\mathbb{L}^s(\mathbf{Z})$. Tensor product of chain complexes endows $\mathbb{L}$ with the structure of a ring spectrum (in fact, an E_∞-ring spectrum). Let X be a space and let ζ be an oriented spherical fibration (of dimension n) over X whose classifying map factors through $\mathbf{Z} \times \mathrm{BPL}$. The above argument shows that the local system

$$(x \in X) \mapsto \Sigma^{-n}(\zeta(x) \wedge \mathbb{L})$$

is equivalent to the constant local system taking the value $\mathbb{L}$. In particular, this local system has a canonical global section; let us denote it by α.

Let us identify the field $\mathbf{Q}$ of rational numbers with the corresponding Eilenberg-MacLane spectrum. Since ζ is oriented, the local system

$$(x \in X) \mapsto \Sigma^{-n}(\zeta(x) \wedge \mathbf{Q})$$

is (canonically) equivalent to the constant local system taking the value $\mathbf{Q}$. Let $\mathbb{L}_{\mathbf{Q}} = \mathbb{L}\mathbf{Q}$ denote the rationalization of $\mathbb{L}$. Smashing with $\mathbb{L}$, we obtain a trivialization of the local system

$$(x \in X) \mapsto \Sigma^{-n}(\zeta(x) \wedge \mathbb{L}_{\mathbf{Q}}).$$

Under this equivalence, α determines a section of the constant local system taking the value $\mathbb{L}_{\mathbf{Q}}$. That is, we can identify α with an (invertible) element $l(\zeta)$ in the cohomology ring $(\mathbb{L}_{\mathbf{Q}})_0(X)$. This element depends naturally on the pair (X, ζ) (together with the orientation and PL-structure on ζ). We may therefore regard $l(\zeta)$ as a *characteristic class* of oriented PL bundles ζ. It measures the discrepancy between two $\mathbb{L}_{\mathbf{Q}}$-orientations of ζ: one coming from the $\mathbb{L}$-orientation defined above, and the other from the $\mathbf{Q}$-orientaiton on ζ.

Using our calculation of the homotopy groups of $\mathbb{L}^q(\mathbf{Z})$, we can be more explicit. We have seen that there is an isomorphism of commutative rings $\pi_* \mathbb{L} \simeq \mathbf{Q}[t^{\pm 1}]$, where t lies in degree -4. We therefore have a homotopy equivalence of spectra $\mathbb{L}_{\mathbf{Q}} \simeq \prod_{k \in \mathbf{Z}} \Sigma^{-4k} \mathbf{Q}$, and therefore an isomorphism of graded rings

$$(\mathbb{L}_{\mathbf{Q}})_*(X) \simeq \prod_{k \in \mathbf{Z}} \mathrm{H}^{*+4k}(X; \mathbf{Q}) \simeq \mathrm{H}^*(X; \mathbf{Q})[[t]]$$

In particular, we can identify $(\mathbb{L}_{\mathbf{Q}})_0(X)$ with the ring $\prod_{k\geq 0} \mathrm{H}^{4k}(X;\mathbf{Q})$, so that $l(\zeta)$ can be written as a formal sum

$$l_0(\zeta) + l_1(\zeta) + \cdots$$

where $l_k(\zeta) \in \mathrm{H}^{4k}(X;\zeta)$.

Now suppose that M is a compact closed oriented PL manifold of dimension n. Then M has orientations with respect to both $\mathbf{Q}$ and the L-theory spectrum $\mathbb{L}$, giving us fundamental classes

$$[M]_{\mathbf{Q}} \in \mathrm{H}_n(X;\mathbf{Q}) \qquad [M]_{\mathbb{L}} \in \mathbb{L}_n(M).$$

Let us identify $[M]_{\mathbf{Q}}$ and $[M]_{\mathbb{L}}$ with their images in $(\mathbb{L}_{\mathbf{Q}})_n(M)$. We then have

$$[M]_{\mathbf{Q}} = l(\zeta)[M]_{\mathbb{L}}$$

where ζ denotes the normal fibration of M. The projection map $p : M \to *$ induces a pushforward

$$p_* : (\mathbb{L}_{\mathbf{Q}})_n(M) \to (\mathbb{L}_{\mathbf{Q}})_n(*) = \begin{cases} \mathbf{Q} & \text{if } n = 4k \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $[M]_{\mathbb{L}}$ is represented by the constant sheaf on M taking the value $\mathbf{Z}$, regarded as a Poincare object of $\mathrm{Shv}_{\mathrm{const}}(M)$, so that $p_*([M]_{\mathbb{L}})$ can be represented by the cochain complex $C^*(M;\mathbf{Z})$, regarded as a Poincare object of $\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}$. If $n = 4k$, then the isomorphism $(\mathbb{L}_{\mathbf{Q}})_n \simeq \mathbf{Q}$ carries $p_*(M)$ to the signature σ_M of the manifold M. This proves an abstract version of the Hirzebruch signature formula: we have

$$\sigma_M = p_*([M]_{\mathbb{L}}) \simeq p_*(l(\zeta)[M]_{\mathbf{Q}}) = l(\zeta)_k[M],$$

where $l(\zeta)_k$ denotes the kth term in the characteristic class $l(\zeta)$ described above.