# Orientations of L-Theory (Lecture 23)

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In the last lecture, we introduced the $L$-theory spectra $\mathbb{L}^q(X,\zeta,R)$ and $\mathbb{L}^s(X,\zeta,R)$, where $R$ is an $A_\infty$-ring with involution, $X$ is a finite polyhedron, and $\zeta$ is a spherical fibration on $X$. When $\zeta$ is trivial, these spectra are given simply by $X \wedge \mathbb{L}^q(R)$ and $X \wedge \mathbb{L}^s(R)$, respectively. In general, they depend on the spherical fibration $\zeta$. However, our excision argument generalizes to show that $\mathbb{L}^q(X,\zeta,R)$ is given by the homotopy colimit

$$\varinjlim_{\tau \in T} \mathbb{L}(\mathrm{LMod}_R^{\mathrm{fp}}, \zeta(\tau) \wedge Q^q)$$

where $T$ denotes any triangulation of $X$. In other words, the homotopy groups of $\mathbb{L}^q(X,\zeta,R)$ are given by the homology of $X$ with coefficients in a local system of spectra, given by $(x \in X) \mapsto \mathbb{L}(\mathrm{LMod}_R^{\mathrm{fp}}, \zeta(x) \wedge Q^q)$. This raises the following general question:

**Question 1.** Let $\mathcal{C}$ be a stable $\infty$-category equipped with a nondegenerate functor $Q$, and let $E$ be an invertible spectrum. What is the relationship between the $L$-theory spectra $\mathbb{L}(\mathcal{C},Q)$ and $\mathbb{L}(\mathcal{C}, E \wedge Q)$?

In the situation of Question 1, we can write $E \simeq S^{-n}$ for some integer $n$. We have seen that there is a canonical isomorphism $L_k(\mathcal{C},\Omega^n Q) = L_{k+n}(\mathcal{C},Q)$, suggesting that we should have an equivalence of $L$-theory spectra $\mathbb{L}(\mathcal{C},\Omega^n Q) \simeq \Omega^n \mathbb{L}(\mathcal{C},Q)$. In other words, we have a homotopy equivalence

$$\theta_E : \mathbb{L}(\mathcal{C}, E \wedge Q) \simeq E \wedge \mathbb{L}(\mathcal{C},Q).$$

For our purposes, we need to know this not just for an individual invertible spectrum $E$, but in the case where $E$ ranges over the fibers of some spherical fibration. It is therefore important that our analysis be functorial with respect to automorphisms of $E$. In fact, it is not possible to choose $\theta_E$ to be functorial with respect to all automorphisms of $E$. However, we will show that it can be chosen to depend naturally on automorphisms which are of geometric origin.

**Definition 2.** Let $M$ be PL manifold, and let $\underline{S}$ denote the local system of spectra on $M$ taking the constant value $S$ (where $S$ is the sphere spectrum). The Verdier dual $\mathbb{D}(\underline{S})$ is a spherical fibration over $M$. We will denote the *inverse* of this spherical fibration by $\zeta_M$. We refer to $\zeta_M$ as the *normal spherical fibration of* $M$. Unwinding the definitions, it can be described by the formula

$$\zeta_M(x) = (\Sigma^\infty(M/M - \{x\}))^{-1}.$$

There is a canonical map of spectra $S \to \Gamma(M;\underline{S})$. If $M$ is compact, this dualizes to give a map

$$\Gamma(M;\mathbb{D}\underline{S}) \simeq \mathbb{D}\Gamma(M;\underline{S}) \to S.$$

This map gives a point in the zeroth space of the spectrum

$$\mathrm{Mor}_{\mathrm{Sp}}(\varprojlim_{\tau \in T}(\mathbb{D}\underline{S})(\tau), S) \simeq \varinjlim_{\tau \in T} \zeta_M(\tau)$$

where $T$ denotes some triangulation of $M$. We will denote this point by $[M]$ and refer to it as the *fundamental class* of $M$.

More generally, if $M$ is a PL manifold with boundary, we let $\zeta_M$ denote the local system of spectra on $M$ obtained by extending the normal spherical fibration from the interior of $M$ (note that the interior of $M$ is homotopy equivalent to $M$, so there exists an essentially unique extension). In this case, we have a fundamental class

$$[M] \in \Omega^{\infty}(\varinjlim_{\tau \in T} \begin{cases} \zeta_M(\tau) & \text{if } \tau \nsubseteq \partial M \\ 0 & \text{otherwise.} \end{cases}$$

Let us now fix a PL manifold with boundary $M$. Let $\mathcal{C}$ be a stable $\infty$-category equipped with a nondegenerate quadratic functor $Q$. For each triangulation $T$ of $M$, let

$$Q_{\zeta_M, T} : \mathrm{Shv}_T(M, \partial M; \mathcal{C})^{op} \to \mathrm{Sp}$$

be given by the formula

$$\varinjlim_{\tau} \begin{cases} Q_{\zeta_M, T}(\mathcal{F}(\tau)) \wedge \zeta_M(\tau) & \text{if } \tau \nsubseteq \partial M \\ 0 & \text{otherwise.} \end{cases}$$

Let $C \in \mathcal{C}$ be an object, and let $\underline{C}$ denote the constant sheaf on $M$ with taking the value $C$ (which we will identify with its image in $\mathrm{Shv}_T(M, \partial M; \mathcal{C})$). We then obtain a homotopy equivalence

$$Q_{\zeta_M, T}(\underline{C}) \simeq Q(C) \wedge \varinjlim_{\tau \in T} \begin{cases} \zeta_M(\tau) & \text{if } \tau \nsubseteq \partial M \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the fundamental class $[M]$ determines a map

$$Q(C) \to Q_{\zeta_M, T}(\underline{C}),$$

which we will denote by $q \mapsto q_{[M]}$. This constuction carries Poincare objects to Poincare objects, and induces a map of $L$-theory spectra

$$\Phi : \mathbb{L}(\mathcal{C}, Q) \to \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(M, \partial M; \mathcal{C}); Q_{\zeta_M})$$

(where $Q_{\zeta_M}$ denotes the amalgamation of the quadratic functors $Q_{\zeta_M, T}$ where $T$ ranges over all triangulations of $M$).

**Example 3.** Let $M$ be a piecewise linear disk. For every point $x$ in the interior of $M$, we have a canonical homotopy equivalence of pairs $(M, \partial M) \to (M, M - \{x\})$. Consequently, $\zeta_M$ is canonically equivalent to the constant sheaf taking the value $E$, where $E^{-1} = \Sigma^{\infty}(M / \partial M)$. It follows that $\mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(M, \partial M; \mathcal{C}), Q_{\zeta})$ is given by $(M, \partial M) \wedge \mathbb{L}(\mathcal{C}, E \wedge Q) \simeq E^{-1} \wedge \mathbb{L}(\mathcal{C}, E \wedge Q)$. We may therefore identify $\Phi$ with a map of spectra $E \wedge \mathbb{L}(\mathcal{C}, Q) \to \mathbb{L}(\mathcal{C}, E \wedge Q)$.

Suppose $M \simeq \Delta^n$. Then $\Phi$ determines maps $L_{k+n}(\mathcal{C}, Q) \to L_k(\mathcal{C}, \Omega^n Q)$, which can be identified with the shift isomorphisms defined earlier. It follows that $\Phi$ is a homotopy equivalence whenever $M$ is a piecewise linear disk.

The construction of $\Phi$ is functorial with respect to piecewise linear homeomorphisms of the PL manifold $M$.

Let us now introduce some terminology to describe the situation more systematically.

Let $X$ be a polyhedron. A *closed $n$-disk bundle* over $X$ is a map of polyhedra $q : D \to X$ such that every point $x \in X$ has an open neighborhood $U$ for which there is a PL homeomorphism $q^{-1}U \simeq U \times \Delta^n$ (which commutes with the projection to $U$).

There is a canonical bijection between isomorphism classes of closed $n$-disk bundles over $X$ and homotopy classes of maps $X \to B\,\mathrm{Disk}(n)$, where $B\,\mathrm{Disk}(n)$ denotes the classifying space of the (simplicial) group $\mathrm{Disk}(n)$ of PL homeomorphisms of $\Delta^n$.

The disjoint union $\coprod_n B\,\mathrm{Disk}(n)$ is equipped with a multiplication which is associative up to coherent homotopy, classifying the formation of products of closed $n$-disk bundles. We can describe the group completion of $\coprod_n B\,\mathrm{Disk}(n)$ as a product $\mathbf{Z} \times \mathrm{BPL}$, where BPL is the direct limit $\varinjlim_n B\,\mathrm{Disk}(n)$.

Let $\mathrm{Pic}(S)$ denote the classifying space for invertible spectra (so that homotopy classes of maps $X \to \mathrm{Pic}(S)$ correspond to equivalence classes of spherical fibrations over $X$). Every closed disk bundle $q : D \to X$ has an associated spherical fibration, given by $x \mapsto \Sigma^\infty(D_x / \partial D_x)$. This construction determines a map $\coprod_n B\,\mathrm{Disk}(n) \to \mathrm{Pic}(S)$, which is multiplicative up to coherent homotopy and therefore extends to a map $\mathbf{Z} \times \mathrm{BPL} \to \mathrm{Pic}(S)$.

Fix $(\mathcal{C}, Q)$ as above. Over the space $\mathrm{Pic}(S)$, we have two local systems of spectra: one given by the formula $E \mapsto \mathbb{L}(\mathcal{C}, E \wedge Q)$ and one given by the formula $E \mapsto E \wedge \mathbb{L}(\mathcal{C}, Q)$. The above analysis implies that these two local systems are canonically equivalent when restricted to $\mathbf{Z} \times \mathrm{BPL}$. This proves the following:

**Proposition 4.** *Let $X$ be a finite polyhedron with triangulation $T$, $\mathcal{C}$ a stable $\infty$-category equipped with a nondegenerate quadratic functor $Q$, and $\zeta$ a spherical fibration on $X$, classified by a map $X \to \mathrm{Pic}(S)$. Suppose that this classifying map factors through $\mathbf{Z} \times \mathrm{BPL}$ (that is, that the spherical fibration $\zeta$ arises from a closed disk bundle, at least stably). Then there is a homotopy equivalence (depending canonically on the factorization)*

$$\mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q_\zeta) \simeq \varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}(\mathcal{C}, Q).$$

We can also make the analysis of the preceding discussion read in a different way. Let us suppose that $\mathcal{C} = \mathcal{D}^{\mathrm{fp}}(\mathbf{Z})$ is the $\infty$-category of perfect complexes of $\mathbf{Z}$-modules, and let $Q$ be either $Q^q$ or $Q^s$. Then $Q$ is a spectrum valued functor which factors through the $\infty$-category of $\mathbf{Z}$-module spectra. It follows that for every spectrum $E$, we can write $E \wedge Q \simeq (E \wedge \mathbf{Z}) \wedge_{\mathbf{Z}} Q$, so that $E \wedge Q$ depends only on the generalized Eilenberg-MacLane spectrum $E \wedge \mathbf{Z}$. Let $\zeta$ be a spherical fibration on a polyhedron $X$, and suppose that $\zeta$ assigns to each point $x \in X$ a spectrum $\zeta(x)$ which is homotopy equivalent to $\Sigma^n S$. Suppose further that $\zeta$ is orientable. A choice of orientation determines a canonical homotopy equivalence of each $\zeta(x) \wedge \mathbf{Z}$ with $\Sigma^n \mathbf{Z}$, and therefore a natural isomorphism $Q_\zeta \simeq \Sigma^n Q$. It follows that we obtain a canonical homotopy equivalence

$$\varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}(\mathcal{C}, Q) \simeq \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q_\zeta) \simeq \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), \Sigma^n Q) \simeq \Sigma^n \mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q) \simeq \Sigma^n (X \wedge \mathbb{L}(\mathcal{C}, Q)).$$

This proves:

**Proposition 5.** *If $\zeta$ is an oriented spherical fibration (of dimension $n$) on $X$ classified by a map $X \to \mathrm{Pic}(S)$ which factors through $\mathbf{Z} \times \mathrm{BPL}$, then we have homotopy equivalences (depending canonically on the choice of factorization)*

$$\varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}^q(\mathbf{Z}) \simeq \Sigma^n (X \wedge \mathbb{L}^q(\mathbf{Z}))$$

$$\varinjlim_{\tau \in T} \zeta(\tau) \wedge \mathbb{L}^s(\mathbf{Z}) \simeq \Sigma^n (X \wedge \mathbb{L}^s(\mathbf{Z}))$$

**Remark 6.** Proposition 5 can be interpreted as saying that every orientable PL bundle is *oriented* with respect to the ring spectrum $\mathbb{L}^s$. We will return to this point in the next lecture.