Assembly (Lecture 22)

March 24, 2011

Let R be an A_∞-ring and let X be a connected finite polyhedron with a triangulation T. In the last lecture, we defined a subcategory $\mathrm{Shv}^0_T(X;\mathrm{LMod}^{\mathrm{fp}}_R) \subseteq \mathrm{Shv}_T(X;\mathrm{LMod}^{\mathrm{fp}}_R)$. Moreover, we showed that the quotient

$$\mathrm{Shv}_T(X;\mathrm{LMod}^{\mathrm{fp}}_R)/\,\mathrm{Shv}^0_T(X;\mathrm{LMod}^{\mathrm{fp}}_R$$

can be identified with the ∞-category $(\mathrm{RMod}^{\mathrm{fp}}_{R'})^{op} \simeq \mathrm{LMod}^{\mathrm{fp}}_{R'}$ of finitely presented R'-module spectra, where $R' \simeq R \wedge \Omega(X)$ is the A_∞-ring whose modules are local systems of R-modules on X.

Our goal in this lecture is to study quadratic functors on $\mathrm{Shv}_T(X;\mathrm{LMod}^{\mathrm{fp}}_R)$ which descend to the quotient category.

Definition 1. Let X be a spectrum. We say that X is *invertible* if there exists another spectrum Y and a homotopy equivalence $X \wedge Y \simeq S$, where S is the sphere spectrum. One can show that a spectrum X is invertible if and only if $X \simeq \Sigma^n S$ for some integer n. We let $\mathrm{Sp}^{\mathrm{inv}}$ denote the full subcategory of Sp spanned by the invertible spectra.

Let X be a space (for now, let's say a polyhedron). A *spherical fibration* over X is a locally constant sheaf on X with values in $\mathrm{Sp}^{\mathrm{inv}}$.

Example 2. Let M be a piecewise linear manifold of dimension n, and let $\mathcal{F}$ be the constant sheaf of spectra taking the value S. Then the Verdier dual $\mathbb{D}(\mathcal{F})$ is a spherical fibration over M. For every point $x \in M$, we have seen that the stalk $x^*\mathbb{D}(\mathcal{F})$ can be described as the suspension spectrum of the homotopy quotient $M/M - \{x\}$, which is (noncanonically) homotopy equivalent to an n-sphere.

Now suppose that R is an A_∞-ring equipped with an involution σ. Let $Q : (\mathrm{LMod}^{\mathrm{fp}}_R)^{op} \to \mathrm{Sp}$ denote either of the quadratic functors Q^q or Q^s. Let $\zeta : T \to \mathrm{Sp}^{\mathrm{inv}}$ be a spherical fibration on X. We define a quadratic functor $Q_\zeta : \mathrm{Shv}_T(X;\mathrm{LMod}^{\mathrm{fp}}_R)^{op} \to \mathrm{Sp}$ by the formula

$$Q_\zeta(\mathcal{F}) = \varinjlim_{\tau\in T} \zeta(\tau) \wedge Q(\mathcal{F}).$$

Since ζ is a constant functor locally on X, the work of the previous lectures shows that Q_ζ is a nondegenerate quadratic functor on $\mathrm{Shv}_T(X;\mathrm{LMod}^{\mathrm{fp}}_R)^{op}$. In particular, we obtain a "twisted" Verdier duality functor $\mathbb{D}_\zeta : \mathrm{Shv}_T(X;\mathrm{LMod}^{\mathrm{fp}}_R)^{op} \to \mathrm{Shv}_T(X;\mathrm{LMod}^{fp}_R)$, given by

$$\mathbb{D}_\zeta(\mathcal{F}) = \zeta \wedge \mathbb{D}(\mathcal{F})$$

(where $\mathbb{D}$ denotes the standard Verdier duality functor discussed earlier).

Lemma 3. *The subcategory* $\mathrm{Shv}^0_T(X;\mathrm{LMod}^{\mathrm{fp}}_R)$ *is stable under the twisted Verdier duality functor* $\mathbb{D}_\zeta$.

Proof. For each simplex $\tau \in T$, define $\mathcal{F}_\tau \in \mathrm{Shv}_T(X;\mathrm{LMod}^{\mathrm{fp}}_R)$ by the formula

$$\mathcal{F}_\tau(\sigma) = \begin{cases} R & \text{if } \sigma \subseteq \tau \\ 0 & \text{otherwise.} \end{cases}$$

Whenever $\tau' \subseteq \tau$, we have a canonical map $\mathcal{F}_\tau \to \mathcal{F}_{\tau'}$. Let us denote the fiber of this map by $\mathcal{F}_{\tau,\tau'}$. We saw in the previous lecture that $\mathrm{Shv}^0_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$ is generated by the objects $\mathcal{F}_{\tau,\tau'}$. It will therefore suffice to show that each $\mathbb{D}_\zeta \mathcal{F}_{\tau,\tau'}$ belongs to $\mathrm{Shv}^0_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$.

If $\tau' \subset \tau'' \subset \tau$, then we have a fiber sequence

$$\mathcal{F}_{\tau,\tau''} \to \mathcal{F}_{\tau,\tau'} \to \mathcal{F}_{\tau'',\tau'} .$$

We may therefore reduce to proving the lemma for the pairs (τ, τ'') and (τ'', τ'). Suppose that $\tau \simeq \Delta^n$ and that τ' is a face of τ; let K denote the closure of $\partial\,\tau - \tau'$. Since τ is contractible, ζ is constant on τ; it will therefore suffice to show that $\mathbb{D}\,\mathcal{F}_{\tau,\tau'}$ belongs to $\mathrm{Shv}^0_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$. A simple calculation shows that $\Sigma^{-n}\mathbb{D}\,\mathcal{F}_{\tau,\tau'}$ is given by the formula

$$\sigma \mapsto \begin{cases} R & \text{if } \sigma \subseteq \tau, \sigma \nsubseteq K \\ 0 & \text{otherwise.} \end{cases}$$

We can choose a different triangulation of X for which τ and K are simplices of the triangulation, in which case the sheaf abvoe is given by $\mathcal{F}_{\tau,K}$, and therefore belongs to $\mathrm{Shv}^0_{\mathrm{const}}(T; \mathrm{LMod}^{\mathrm{fp}}_R)$. □

Using the formalism of Lecture 8, we see that Q_ζ descends to give a quadratic functor Q_{lc} on $\mathrm{LMod}^{\mathrm{fp}}_{R'}$. When $Q = Q^s$, we will denote this functor by Q^s_{lc}; when $Q = Q^q$, we will denote this functor by Q^q_{lc}.

Notation 4. Let X be a finite polyhedron, R an A_∞-ring with involution, and ζ a spherical fibration on X. We let $\mathbb{L}^q(X, \zeta, R)$ denote the L-theory spectrum of the pair $(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}^{\mathrm{fp}}_R), Q^q_\zeta)$ and $\mathbb{L}^{vq}(X, \zeta, R)$ the L-theory spectrum of the pair $(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}^{\mathrm{fp}}_R)/\,\mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}^{\mathrm{fp}}_R), Q^q_{\mathrm{lc}})$. Similarly, we let $\mathbb{L}^s(X, \zeta, R)$ denote the L-theory spectrum of the pair $(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}^{\mathrm{fp}}_R), Q^s_\zeta)$ and $\mathbb{L}^{vs}(X, \zeta, R)$ the L-theory spectrum of the pair $(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}^{\mathrm{fp}}_R)/\,\mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}^{\mathrm{fp}}_R), Q^s_{\mathrm{lc}})$. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{L}^q(X, \zeta, R) & \longrightarrow & \mathbb{L}^s(X, \zeta, R) \\ \downarrow & & \downarrow \\ \mathbb{L}^{vq}(X, \zeta, R) & \longrightarrow & \mathbb{L}^{vs}(X, \zeta, R) \end{array}$$

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Using the fact that Q^q and Q^s have the same associated bilinear functor B, we deduce that Q^s_{lc} and Q^q_{lc} have the same associated bilinear functor B_{lc}. Let us try to describe this bilinear functor more explicitly. Set $Q = Q^s$, so that R is a Poincare object of $\mathrm{LMod}^{\mathrm{fp}}_R$. Let $x \in X$ be our chosen base point. We will assume that x is a vertex of the triangulation T, and suppose that the invertible spectrum ζ_x is homotopy equivalent to the sphere spectrum (something that can always be achieved by an appropriate shift). Let $x_*(R) \in \mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$ denote the skyscraper sheaf with stalk R at the point x (and vanishing elsewhere). Then $x_*(R)$ has the structure of a Poincare object of $\mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$. It follows that the image of $x_*(R)$ in the ∞-category $\mathrm{LMod}^{\mathrm{fp}}_{R'}$ has the structure of a Poincare object of $\mathrm{LMod}^{\mathrm{fp}}_{R'}$. By construction, this image can be identified with R' itself. Arguing as in Lecture 10, we deduce that R' is equipped with an involution σ, and that the bilinear functor $B_{\mathrm{lc}} : (\mathrm{LMod}^{\mathrm{fp}}_R)^{op} \times (\mathrm{LMod}^{\mathrm{fp}}_R)^{op} \to \mathrm{Sp}$ is given by $(M, N) \mapsto \mathrm{Mor}_{R'-R'}(M \wedge N, R')$.

Remark 5. To be even more explicit, one would like to describe the involution on the A_∞-ring R'. This involves a mixture of three ingredients:

(*i*) The given involution on R.

(*ii*) The involution of the loop space $\Omega(X) \simeq P_{x,x}$, given by reading each path in the opposite direction.

(*iii*) The nontriviality of the spherical fibration ζ

Suppose for example that R is connective, so that R' is connective and we have a canonical isomorphism of associative rings $\pi_0 R' \simeq (\pi_0 R)[\pi_1 X]$. Then the involution on $\pi_0 R'$ is given by

$$\sum_{g \in \pi_1 X} \lambda_g g \mapsto \sum_{g \in \pi_1 X} \epsilon(g)\sigma(\lambda_g) g^{-1}$$

where σ denotes the underlying involution on $\pi_0 R$ and $\epsilon : \pi_1 X \to \pm 1$ is the obstruction to choosing an orientation of the spherical fibration ζ.

Now armed with our description of B_{lc}, let us try to to describe Q_{lc} in the special case where $Q = Q^q$. Let us regard Q as a covariant functor $\mathrm{RMod}_R^{\mathrm{fp}} \to \mathrm{Sp}$, given by $Q(M) = B(M, M)_{h\Sigma_2}$. Then Q_ζ corresponds to the covariant functor $\mathrm{coShv}_T(X; \mathrm{RMod}_R^{\mathrm{fp}}) \to \mathrm{Sp}$ given by

$$Q_\zeta(\mathcal{F}) = \varinjlim_{\tau \in T} \zeta(\tau) \wedge B(\mathcal{F}(\tau), \mathcal{F}(\tau))_{h\Sigma_2}.$$

We can extend this to a functor

$$\widehat{Q}_\zeta(\mathcal{F}) : \mathrm{Ind}(\mathrm{coShv}_T(X; \mathrm{RMod}_R^{\mathrm{fp}})) \simeq \mathrm{coShv}_T(X; \mathrm{RMod}_R) \to \mathrm{Sp}$$

which commutes with filtered colimits; this is again given by the formula

$$\widehat{Q}_\zeta(\mathcal{F}) = \varinjlim_{\tau} \zeta(\tau) \wedge (\mathcal{F}(\tau) \wedge_R \mathcal{F}(\tau))_{h\Sigma_2}$$

The quadratic functor

$$Q_{\mathrm{lc}} : (\mathrm{LMod}_{R'}^{\mathrm{fp}})^{op} \simeq \mathrm{RMod}_{R'}^{\mathrm{fp}} \to \mathrm{Sp}$$

is given by composing $\widehat{Q}_\zeta$ with the fully faithful embedding

$$\theta : \mathrm{RMod}_{R'}^{\mathrm{fp}} \subseteq \mathrm{RMod}_{R'} \simeq \mathrm{coShv}_{\mathrm{lc}}(X : \mathrm{RMod}_R) \subseteq \mathrm{coShv}_T(X; \mathrm{RMod}_R).$$

It follows that the polarization B_{lc} is given by composing θ with the map $\widehat{B}_\zeta$, given by

$$(\mathcal{F}, \mathcal{G}) \mapsto \varinjlim_{\tau} \zeta(\tau) \wedge (\mathcal{F}(\tau) \wedge_R \mathcal{G}(\tau)).$$

We deduce that the natural map $B_{\mathrm{lc}}(M, M)_{h\Sigma_2} \to Q^q_{\mathrm{lc}}$ is an equivalence. This proves the following:

Proposition 6. *Let R be an A_∞-ring with involution, let X be a connected finite polyhedron with base point x, let ζ be a spherical fibration over X equipped with a trivialization at x, and let $R' \simeq R \wedge \Omega(X)$ be defined as above. Then we have a canonical homotopy equivalence $\mathbb{L}^{vq}(X, \zeta, R) \simeq \mathbb{L}^q(R')$, where R' is the A_∞-ring with involution described above. In particular, if R is connective, there is a canonical equivalence $\mathbb{L}(\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}) / \mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}), Q^q_{\mathrm{lc}}) \simeq \mathbb{L}^q((\pi_0 R)[\pi_1 X])$, where the involution on the group ring $(\pi_0 R)[\pi_1 X]$ is described in Remark 5.*

The analogous statement is generally not true for symmetric L-theory, because the construction $M \mapsto (M \wedge_R M)^{h\Sigma_2}$ generally does not commute with filtered colimits.

Assume that X is connected with base point x, that ζ is trivial, and that R is connective. In this case, the π-π theorem and Proposition 6 allow us to simplify the commutative diagram appearing in Notation 4:

$$\begin{array}{ccc} X \wedge \mathbb{L}^q(\pi_0 R) & \longrightarrow & X \wedge \mathbb{L}^s(R) \\ \downarrow & & \downarrow \\ \mathbb{L}^q((\pi_0 R)[\pi_1 X]) & \longrightarrow & \mathbb{L}^v_{(X,\zeta)}(R). \end{array}$$

The vertical maps here are referred to as *assembly maps.*

Proposition 7. *The diagram appearing in Notation 4 is a homotopy pullback square of spectra.*

Proof. It suffices to show that we get a homotopy equivalence between the homotopy fibers of the vertical maps. Using localization theorem of Lecture 8, we can identify these homotopy fibers with the L-theory spectra of $\mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}^{\mathrm{fp}}_R)$ with respect to Q^q_ζ and Q^s_ζ, respectively. It will therefore suffice to show that the canonical map $Q^q_\zeta \to Q^s_\zeta$ is an equivalence when evaluated on any $\mathcal{F} \in \mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}^{\mathrm{fp}}_R)$.

Let $U : (\mathrm{LMod}^{\mathrm{fp}}_R)^{op} \to \mathrm{Sp}$ be the homotopy cofiber of the map $Q^q \to Q^s$, so that U is given by the formula $U(M) = \mathrm{Mor}_{R-R}(M \wedge M, R)^{t\Sigma_2}$. Then U is an exact functor. It follows that $U(R)$ has the structure of an R-module spectrum, and that U is given by the formula $U(M) = \mathrm{Mor}_R(M, U(R))$. We deduce that for $\mathcal{F} \in \mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$, the cofiber of the map $Q^q_\zeta(\mathcal{F}) \to Q^s_\zeta(\mathcal{F})$ is given by

$$\varinjlim_{\tau \in T} \zeta(\tau) \wedge U(\mathcal{F}(\tau)) = \mathrm{Mor}_R(\varprojlim_{\tau} \zeta(\tau)^{-1} \wedge \mathcal{F}(\tau), U(R)).$$

If $\mathcal{F} \in \mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}^{\mathrm{fp}}_R)$, then the limit $\varprojlim_\tau \zeta(\tau)^{-1} \wedge \mathcal{F}(\tau)$ vanishes (since it is the spectrum of maps from the locally constant sheaf $\zeta \wedge R$ into $\mathcal{F}$). □