Locally Constant Sheaves (Lecture 21)

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Let X be a finite polyhedron with a triangulation T and let $\mathcal{C}$ be an ∞-category. We will say that T-constructible sheaf $\mathcal{F} : T \to \mathcal{C}$ is *locally constant* if $\mathcal{F}(\tau) \to \mathcal{F}(\tau')$ is invertible whenever $\tau \leq \tau'$. We let $\mathrm{Shv}_{\mathrm{lc}}(X; \mathcal{C})$ denote the full subcategory of $\mathrm{Shv}_T(X; \mathcal{C})$ spanned by the locally constant sheaves. This ∞-category does not depend on the choice of triangulation: if S is a refinement of T, then the pullback functor $\mathrm{Shv}_T(X; \mathcal{C}) \to \mathrm{Shv}_S(X; \mathcal{C})$ induces an equivalence on the full subcategories spanned by the locally constant sheaves.

Let R be an A_∞-ring, fixed for the remainder of this lecture. Our goal is to study *local systems* of R-modules on $\mathcal{C}$: that is, locally constant sheaves on $\mathcal{C}$ with values in the ∞-category of R-modules.

Fix a finite polyhedron X and a triangulation T of X. Let $\mathrm{Shv}_T(X : R)$ denote the ∞-category of T-constructible sheaves on X with values in $\mathrm{LMod}_R^{\mathrm{fp}}$: that is, contravariant functors $T^{op} \to \mathrm{LMod}_R^{\mathrm{fp}}$. The formation of R-linear duals gives a contravariant equivalence of $\mathrm{LMod}_R^{\mathrm{fp}}$ with $\mathrm{RMod}_R^{\mathrm{fp}}$; we may therefore identify $\mathrm{Shv}_T(X; R)$ with the opposite of the ∞-category $\mathrm{coShv}_T(X; \mathrm{RMod}_R^{\mathrm{fp}})$ of T-constructible *cosheaves* with values in $\mathrm{RMod}_R^{\mathrm{fp}}$ (that is, functors $T^{op} \to \mathrm{RMod}_R^{\mathrm{fp}}$). This is contained in the larger ∞-category $\mathrm{coShv}_T(X; \mathrm{RMod}_R)$ of cosheaves with values in RMod_R. In fact, we can identify $\mathrm{coShv}_T(X; \mathrm{RMod}_R)$ with the ∞-category of Ind-objects $\mathrm{Ind}(\mathrm{coShv}_T(X; \mathrm{RMod}_R^{\mathrm{fp}})$.

Let $\mathrm{coShv}_{\mathrm{lc}}(X : \mathrm{RMod}_R)$ denote the full subcategory of $\mathrm{coShv}_T(X; \mathrm{RMod}_R)$ spanned by the locally constant cosheaves (that is, those cosheaves for which $\mathcal{F}(\tau) \to \mathcal{F}(\tau')$ is an equivalence for every $\tau' \subseteq \tau \in T$). Note that $\mathrm{coShv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$ is closed under all limits and colimits in $\mathrm{coShv}_T(X; \mathrm{RMod}_R)$. It follows that the inclusion

$$\mathrm{coShv}_{\mathrm{lc}}(X; \mathrm{RMod}_R) \hookrightarrow \mathrm{coShv}_T(X; \mathrm{RMod}_R)$$

admits both left and right adjoints. We will denote a left adjoint to this inclusion by L. Let $\mathrm{coShv}_T^0(X; \mathrm{RMod}_R)$ denote the full subcategory of $\mathrm{coShv}_T(X; \mathrm{RMod}_R)$ spanned by those objects $\mathcal{F}$ such that $T(\mathcal{F}) \simeq 0$: that is, those objects $\mathcal{F}$ such that $\mathrm{Mor}_{\mathrm{coShv}_T(X; \mathrm{RMod}_R)}(\mathcal{F}, \mathcal{G}) \simeq 0$ whenever $\mathcal{G}$ is locally constant. We let $\mathrm{coShv}_T^0(X; \mathrm{RMod}_R^{\mathrm{fp}})$ denote the intersection $\mathrm{coShv}_T^0(X; \mathrm{RMod}_R) \cap \mathrm{coShv}_T(X; \mathrm{RMod}_R^{\mathrm{fp}})$.

Lemma 1. *The full subcategory $\mathrm{coShv}_T^0(X; \mathrm{RMod}_R)$ is generated (under filtered colimits) by $\mathrm{coShv}_T^0(X; \mathrm{RMod}_R^{\mathrm{fp}})$. Consequently, we have a canonical equivalence*

$$\mathrm{coShv}_T^0(X; \mathrm{RMod}_R) \simeq \mathrm{Ind}\,\mathrm{coShv}_T^0(X; \mathrm{RMod}_R^{\mathrm{fp}}).$$

Proof. Since every object of $\mathrm{coShv}_T^0(X; \mathrm{RMod}_R^{\mathrm{fp}})$ is a compact object of $\mathrm{coShv}_T(X; \mathrm{RMod}_R)$, we get a fully faithful embedding

$$\mathrm{Ind}(\mathrm{coShv}_T^0(X; \mathrm{RMod}_R^{\mathrm{fp}}) \to \mathrm{coShv}_T(X; \mathrm{RMod}_R).$$

Let $\mathcal{C}$ denote the essential image of this embedding; it is a full subcategory of $\mathrm{Shv}_T(X; \mathrm{RMod}_R)$. We clearly have $\mathcal{C} \subseteq \mathrm{coShv}_T^0(X; \mathrm{RMod}_R)$. Let us prove the reverse inclusion. For any object $\mathcal{F} \in \mathrm{coShv}_T^0(X; \mathrm{RMod}_R)$, we can choose a fiber sequence

$$\mathcal{F}' \to \mathcal{F} \stackrel{\alpha}{\to} \mathcal{F}''$$

where $\mathcal{F}' \in \mathcal{C}$ and $\mathrm{Mor}(\mathcal{G}, \mathcal{F}'') \simeq 0$ for every $\mathcal{G} \in \mathcal{C}$ (here $\mathcal{F}'$ is given by the colimit of the filtered diagram of all objects of $\mathrm{coShv}^0_T(X; \mathrm{RMod}^{\mathrm{fp}}_R)$ equipped with a map to $\mathcal{F}$). For every simplex $\tau \in T$, let $\mathcal{F}_\tau \in \mathrm{coShv}_T(X; \mathrm{RMod}_R)$ be given by the formula

$$\mathcal{F}_\tau(\sigma) = \begin{cases} R & \text{if } \sigma \subseteq \tau \\ 0 & \text{otherwise.} \end{cases}$$

For any cosheaf $\mathcal{G}$, we have $\mathrm{Mor}(\mathcal{F}_\tau, \mathcal{G}) \simeq \mathcal{G}(\tau)$. Let $\tau' \subseteq \tau$, and form a cofiber sequence

$$\mathcal{F}_{\tau'} \to \mathcal{F}_\tau \to \mathcal{F}_\tau \,/\, \mathcal{F}_{\tau'} \,.$$

If $\mathcal{G}$ is locally constant, we have $\mathrm{Mor}(\mathcal{F}_\tau \,/\, \mathcal{F}_{\tau'}, \mathcal{G}) \simeq \mathrm{fib}(\mathcal{G}(\tau) \to \mathcal{G}(\tau')) \simeq 0$, so that $\mathcal{F}_\tau \,/\, \mathcal{F}_{\tau'} \in \mathrm{coShv}^0_T(X; \mathrm{RMod}^{\mathrm{fp}}_R)$. It follows that $\mathrm{Mor}(\mathcal{F}_\tau \,/\, \mathcal{F}_{\tau'}, \mathcal{F}'') \simeq 0$, so that $\mathcal{F}''(\tau) \simeq \mathcal{F}''(\tau')$. Since τ and τ' are arbitrary, we deduce that $\mathcal{F}''$ is locally constant. Since $\mathcal{F} \in \mathrm{coShv}^0_T(X; \mathrm{RMod}_R)$, the map α is nullhomotopic. Then $\mathcal{F}$ is a direct summand of $\mathcal{F}'$, and therefore belongs to $\mathcal{C}$ as desired. □

Under the contravariant equivalence of ∞-categories $\mathrm{coShv}_T(X; \mathrm{RMod}^{\mathrm{fp}}_R) \simeq \mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$, the subcategory $\mathrm{coShv}^0_T(X; \mathrm{RMod}^{\mathrm{fp}}_R)$ corresponds to a full subcategory $\mathrm{Shv}^0_T(X : \mathrm{LMod}^{\mathrm{fp}}_R) \subseteq \mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$, which is evident closed under the formation of direct summands. Several lectures ago, we constructed a quotient ∞-category

$$\mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R) / \, \mathrm{Shv}^0_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)$$

as a full subcategory of $\mathrm{Pro}(\mathrm{Shv}_T(X; \mathrm{LMod}^{\mathrm{fp}}_R)) \simeq \mathrm{Ind}(\mathrm{coShv}_T(X; \mathrm{RMod}^{\mathrm{fp}}_R))^{op} \simeq \mathrm{coShv}_T(X; \mathrm{RMod}_R)^{op}$. Unwinding the definitions, we see that this subcategory consists precisely of those objects of the form $L\,\mathcal{F}$, where $\mathcal{F} \in \mathrm{coShv}_T(X; \mathrm{RMod}_R)^{\mathrm{fp}})$. Let us denote this subcategory by $\mathrm{coShv}^{\mathrm{fp}}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$.

We now study the ∞-category $\mathrm{coShv}_{\mathrm{lc}}(X; \mathrm{RMod}_R) \simeq \mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$ in more detail. For simplicity, let us restrict our attention to the case where X is connected. For every point $x \in X$, let $i_x : \{x\} \to X$ denote the inclusion map. Pullback along i_x determines a functor $x^* : \mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R) \to \mathrm{RMod}_R$ (given by evaluation at the unique simplex $\tau \in T$ containing x in its interior). This functor commutes with all limits and colimits. In particular, it admits a left adjoint, which we will denote by $x_+ : \mathrm{RMod}_R \to \mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$. Since x^* commutes with filtered colimits, $x_+(R)$ is a compact object of $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$. Moreover, it is a compact *generator* of $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$: if $\mathcal{F} \in \mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$, then $\mathrm{Mor}(x_+(R), \mathcal{F}) \simeq 0$ if and only if $\mathrm{Mor}_{\mathrm{RMod}_R}(R, x^* \mathcal{F}) = x^* \mathcal{F}$ vanishes. Since X is connected, this is equivalent to the vanishing of *all* stalks of $\mathcal{F}$: that is, to the condition that $\mathcal{F} \simeq 0$. It follows that $\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$ is equivalent to the ∞-category $\mathrm{RMod}_{R'}$, where R' is the A_∞-ring whose underlying spectrum is $\mathrm{Mor}_{\mathrm{Shv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)}(x_+(R), x_+(R)) \simeq \mathrm{Mor}_{\mathrm{RMod}_R}(R, x^* x_+ R) \simeq x^* x_+ R$.

We can describe R' more explicitly. More generally, suppose we are given any pair of points $x, y \in X$. We can form a homotopy pullback diagram of topological spaces (commutative up to canonical homotopy)

$$\begin{array}{ccc} P_{x,y} & \xrightarrow{\phi} & \{x\} \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle x} \\ \{y\} & \xrightarrow{y} & X, \end{array}$$

where $P_{x,y}$ is the path space $\{p : [0,1] \to X : p(0) = x, p(1) = y\}$. Let ϕ^* and ψ^* denote the pullback functors on locally constant sheaves of right R-modules, and let ϕ_+ and ψ_+ be their left adjoints. There is a natural "base-change" isomorphism $y^* x_+ \simeq \psi_+ \phi^*$ of functors from RMod_R to itself. Consequently, $y^* x_+(R)$ is given by ψ_+ of the constant sheaf on $P_{x,y}$ with values in R. This is given by the smash product spectrum $P_{x,y} \wedge R$ (here we regard $P_{x,y}$ as an *unpointed* space) whose homotopy groups are given by the R-homology groups $R_*(P_{x,y})$ of the space $P_{x,y}$. In particular, we have $R' = P_{x,x} \wedge R$. If R is connective, we obtain

$$\pi_0 R' = R_0(P_{x,x}) = \bigoplus_{\eta \in \pi_0 P_{x,x}} \pi_0 R \simeq (\pi_0 R)[\pi_1 X].$$

Let us now describe the full subcategory $\mathrm{coShv}_{\mathrm{lc}}^{\mathrm{fp}}(X; \mathrm{RMod}_R) \subseteq \mathrm{coShv}_{\mathrm{lc}}(X; \mathrm{RMod}_R) \overset{\theta}{\simeq} \mathrm{RMod}_{R'}$. This is a stable subcategory, consisting of those objects of the form $L(\mathcal{F})$, where $\mathcal{F} \in \mathrm{coShv}_T(X; \mathrm{RMod}_R^{\mathrm{fp}})$. Note that $\mathrm{coShv}_T(X; \mathrm{RMod}_R^{\mathrm{fp}})$ is generated, as a stable ∞-category, by objects of the form $\mathcal{F}_{\tau,M}$, where

$$\mathcal{F}_{\tau,M}(\sigma) = \begin{cases} M & \text{if } \sigma \subseteq \tau \\ 0 & \text{otherwise} \end{cases}$$

and M is a finitely presented right R-module. It is therefore generated as a stable ∞-category by objects of the form $\mathcal{F}_{\tau,R} = \mathcal{F}_\tau$. We observe that $L\,\mathcal{F}_\tau \simeq y_+(R)$, where y is any point in the interior of τ: indeed, for any object $\mathcal{G} \in \mathrm{coShv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$ we have

$$\mathrm{Mor}(L\,\mathcal{F}_\tau, \mathcal{G}) \simeq \mathrm{Mor}(\mathcal{F}_\tau, \mathcal{G}) \simeq \mathcal{G}(\tau) \simeq y^* \,\mathcal{G} \simeq \mathrm{Mor}(y_+(R), \mathcal{G}).$$

Let x be our fixed base point of X. Since X is connected, for any point $y \in X$ there is an isomorphism $x_+(R) \simeq y_+(R)$ in $\mathrm{coShv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$ (obtained by choosing a path joining x and y). Consequently, the full subcategory $\mathrm{coShv}_{\mathrm{lc}}^{\mathrm{fp}}(X; \mathrm{RMod}_R) \subseteq \mathrm{coShv}_{\mathrm{lc}}(X; \mathrm{RMod}_R)$ is generated, as a stable subcategory, by the object $x_+(R)$. In particular, it corresponds to the full subcategory

$$\mathrm{RMod}_{R'}^{\mathrm{fp}} \subseteq \mathrm{RMod}_R$$

under the equivalence θ. Passing to opposite ∞-categories, we obtain the following result:

Theorem 2. *Let X be a connected finite polyhedron with base point x, T a triangulation of X, R an A_∞-ring, and $R' = P_{x,x} \wedge R$ the A_∞-ring constructed above. Let $\mathrm{Shv}_T^0(X; \mathrm{LMod}_R^{\mathrm{fp}}) \subseteq \mathrm{Shv}_T(X; \mathrm{LMod}_R^{\mathrm{fp}})$ be defined as above. Then there is a canonical equivalence of ∞-categories*

$$\mathrm{Shv}_T(X; \mathrm{LMod}_R^{\mathrm{fp}}) / \mathrm{Shv}_T^0(X; \mathrm{LMod}_R^{\mathrm{fp}}) \simeq \mathrm{LMod}_{R'}^{\mathrm{fp}}.$$

Let $\mathrm{Shv}_{\mathrm{const}}^0(X; \mathrm{LMod}_R^{\mathrm{fp}}) \simeq \varinjlim_T \mathrm{Shv}_T^0(X; \mathrm{LMod}_R^{\mathrm{fp}}$. Passing to the direct limit over T, we obtain an equivalence

$$\mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}) / \mathrm{Shv}_{\mathrm{const}}^0(X; \mathrm{LMod}_R^{\mathrm{fp}}) \simeq \mathrm{LMod}_{R'}^{\mathrm{fp}}.$$

Warning 3. Working with perfect module spectra in place of finitely presented module spectra, one can construct a fully faithful embedding

$$\mathrm{Shv}_T(X; \mathrm{LMod}_R^{\mathrm{perf}}) / \mathrm{Shv}_T^0(X; \mathrm{LMod}_R^{\mathrm{fp}}) \simeq \mathrm{LMod}_{R'}^{\mathrm{perf}}.$$

This embedding is generally not an equivalence, which is why we have generally confined our attention to the study of finitely presented modules rather than perfect modules.

Definition 4. Let X be a spectrum. We say that X is *invertible* if there exists another spectrum Y and a homotopy equivalence $X \wedge Y \simeq S$, where S is the sphere spectrum. One can show that a spectrum X is invertible if and only if $X \simeq \Sigma^n S$ for some integer n. We let $\mathrm{Sp}^{\mathrm{inv}}$ denote the full subcategory of Sp spanned by the invertible spectra.

Let X be a space (for now, let's say a polyhedron). A *spherical fibration* over X is a locally constant sheaf on X with values in $\mathrm{Sp}^{\mathrm{inv}}$.

Example 5. Let M be a piecewise linear manifold of dimension n, and let $\mathcal{F}$ be the constant sheaf of spectra taking the value S. Then the Verdier dual $\mathbb{D}(\mathcal{F})$ is a spherical fibration over M. For every point $x \in M$, we have seen that the stalk $x^*\mathbb{D}(\mathcal{F})$ can be described as the suspension spectrum of the homotopy quotient $M/M - \{x\}$, which is (noncanonically) homotopy equivalent to an n-sphere.

Now suppose that R is an A_∞-ring equipped with an involution σ. Let $Q : (\mathrm{LMod}_R^{\mathrm{fp}})^{op} \to \mathrm{Sp}$ denote either of the quadratic functors Q^q or Q^s. Let X be a polyhedron equipped with a triangulation T, and let $\zeta : T \to \mathrm{Sp}^{\mathrm{inv}}$ be a spherical fibration on X. We define a quadratic functor $Q_{T,\zeta} : \mathrm{Shv}_T(X; \mathrm{LMod}_R^{\mathrm{fp}})^{op} \to \mathrm{Sp}$ by the formula

$$Q_{T,\zeta}(\mathcal{F}) = \varinjlim_{\tau \in T} \zeta(\tau) \wedge Q(\mathcal{F}).$$

Since ζ is a constant functor locally on X, the work of the previous lectures shows that $Q_{T,\zeta}$ is a nondegenerate quadratic functor on $\mathrm{Shv}_T(X; \mathrm{LMod}_R^{\mathrm{fp}})^{op}$. Passing to the limit over T, we obtain a nondegenerate quadratic functor

$$Q_\zeta : \mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}})^{op} \to \mathrm{Sp}\,.$$

In particular, we obtain a "twisted" Verdier duality functor $\mathbb{D}_\zeta : \mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}})^{op} \to \mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{fp})$, given by

$$\mathbb{D}_\zeta(\mathcal{F}) = \zeta \wedge \mathbb{D}(\mathcal{F})$$

(where $\mathbb{D}$ denotes the standard Verdier duality functor discussed earlier).

In the next lecture, we will apply the paradigm of Lecture 8 to the cofiber sequence of stable ∞-categories

$$\mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}) \to \mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}) \to \mathrm{Shv}_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}}) / \mathrm{Shv}^0_{\mathrm{const}}(X; \mathrm{LMod}_R^{\mathrm{fp}})$$

and the quadratic functor Q_ζ.