L-Groups of Polyhedra (Lecture 20)

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Let $\mathcal{C}$ be a stable ∞-category equipped with a nondegenerate quadratic functor $Q : \mathcal{C}^{op} \to \mathrm{Sp}$. Let X be a finite polyhedron. In the last lecture, we proved that Q determines a nondegenerate quadratic functor $\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})^{op} \to \mathrm{Sp}$. Let us denote this functor by Q_X, to emphasize its dependence on X. We let $L(X; \mathcal{C}, Q)$ denote the L-theory space of the pair $(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q_X)$.

Example 1. When X consists of a single point, we have $L(X; \mathcal{C}, Q) \simeq L(\mathcal{C}, Q)$.

Remark 2. Let $f : X \to Y$ be a map of finite polyhedra, and choose triangulations S and T of X and Y such that f is linear on each simplex. Let $\mathcal{F} \in \mathrm{Shv}_S(X; \mathcal{C})$. Then we have a canonical map

$$Q_S(\mathcal{F}) \simeq \varinjlim_{\sigma \in S} Q(\mathcal{F}(\sigma)) \simeq \varinjlim_{\tau \in T} \varinjlim_{f(\sigma) = \tau} Q(\mathcal{F}(\sigma)) \to \varinjlim_{\tau \in T} Q(\varprojlim_{f(\sigma)=\tau} \mathcal{F}(\sigma)) = Q_T(f_* \mathcal{F}).$$

Taking the direct limit over triangulations, we obtain a natural transformation $Q_X \to Q_Y \circ f_*$. This natural transformation induces a natural transformation

$$f_* \circ \mathbb{V}\mathbb{D} \to \mathbb{V}\mathbb{D} \circ f_*$$

which we showed to be an equivalence in the previous lecture.

Consequently, the pushforward functor f_* carries quadratic objects of $\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$ to quadratic objects of $\mathrm{Shv}_{\mathrm{const}}(Y; \mathcal{C})$ and carries Poincare objects to Poincare objects. We obtain a map of classifying spaces $\mathrm{Poinc}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q_X) \to \mathrm{Poinc}(\mathrm{Shv}_{\mathrm{const}}(Y; \mathcal{C}), Q_Y)$. The same reasoning gives a map of simplicial spaces

$$\mathrm{Poinc}(\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}), Q_X)_\bullet \to \mathrm{Poinc}(\mathrm{Shv}_{\mathrm{const}}(Y; \mathcal{C}), Q_Y)_\bullet$$

hence a map of L-theory spaces

$$L(X; \mathcal{C}, Q) \to L(Y; \mathcal{C}, Q).$$

In other words, $L(X; \mathcal{C}, Q)$ depends functorially on X.

We now study the functor $X \mapsto L(X; \mathcal{C}, Q)$.

Lemma 3. *Let $n \geq 0$ be an integer, and suppose that $f, g : X \to Y$ are homotopic PL maps of finite polyhedra. Then f and g induce the same map $L_n(X; \mathcal{C}, Q) \to L_n(Y; \mathcal{C}, Q)$.*

Proof. Replacing Q by $\Sigma^{-n} Q$, we can reduce to the case $n = 0$. Let $(\mathcal{F}, q)$ be a Poincare object of $\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$. We wish to show that the Poincare objects $f_* \mathcal{F}$ and $g_* \mathcal{F}$ are cobordant. Choose a PL map $h : X \times [0,1] \to Y$ which is a homotopy from f to g. Let $i_0 : X \simeq X \times \{0\} \hookrightarrow X \times [0,1]$ be the canonical map, and define i_1 similarly. Since the pushforward functor h_* carries cobordisms to cobordisms, it will suffice to show that $i_{0,*} \mathcal{F}$ and $i_{1,*} \mathcal{F}$ are cobordant as Poincare objects of $\mathrm{Shv}_{\mathrm{const}}(X \times [0,1]; \mathcal{C})$. It now suffices to observe that a bordism between these objects is given by $p^* \mathcal{F}$, where $p : X \times [0,1] \to X$ denotes the projection. $\square$

From Lemma 3 we immediately deduce the following consequence:

Proposition 4. *Let $f : X \to Y$ be a PL homotopy equivalence between finite polyhedra. Then f induces a homotopy equivalence $L(X; \mathcal{C}, Q) \to L(Y; \mathcal{C}, Q)$.*

Let Poly denote the category whose objects are finite polyhedra and whose morphisms are PL maps. The construction $X \mapsto L(X; \mathcal{C}, Q)$ determines a functor from the category Poly to the ∞-category $\mathcal{S}$ of spaces. It follows from Proposition 4 that this functor factors through $\mathrm{Poly}[W^{-1}]$, where $\mathrm{Poly}[W^{-1}]$ denotes the ∞-category obtained from Poly by formally inverting all homotopy equivalences between finite polyhedra. The ∞-category $\mathrm{Poly}[W^{-1}]$ is equivalent to the full subcategory $\mathcal{S}^{\mathrm{fin}} \subseteq \mathcal{S}$ spanned by those spaces which are homotopy equivalent to a finite polyhedron (or equivalently, to a finite CW complex). We may therefore regard the functor $X \mapsto L(X; \mathcal{C}, Q)$ as defined on the ∞-category $\mathcal{S}^{\mathrm{fin}}$ of finite spaces.

To continue our analysis, it will be convenient to introduce a slight variation on the above construction. Let X be a finite polyhedron, and let $Y \subseteq X$ be a closed subpolyhedron. We then have a fully faithful embedding $i_* : \mathrm{Shv}_{\mathrm{const}}(Y; \mathcal{C}) \to \mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$ which commutes with Verdier duality. It follows that the quotient ∞-category $\mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C}) / \mathrm{Shv}_{\mathrm{const}}(Y; \mathcal{C})$ inherits a nondegenerate quadratic functor. This quotient can be identified with a full subcategory of $\mathrm{Shv}_{\mathrm{const}}(X, Y; \mathcal{C}) \subseteq \mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$: namely, the subcategory spanned by those sheaves $\mathcal{F}$ such that $i^* \mathcal{F} \simeq 0$. (Note that, for any $\mathcal{F} \in \mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$, the ∞-category of sheaves $\mathcal{F}' \in \mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$ equipped with a map $\mathcal{F}' \to \mathcal{F}$ whose cofiber is supported on Y has a final object, given by the extension by zero of $\mathcal{F}|(X - Y)$.) We let $L(X, Y; \mathcal{C}, Q)$ denote the L-theory space of $(\mathrm{Shv}_{\mathrm{const}}(X, Y; \mathcal{C}), Q_X)$. We have seen that there is a fiber sequence of spaces

$$L(Y; \mathcal{C}, Q) \to L(X; \mathcal{C}, Q) \to L(X, Y; \mathcal{C}, Q).$$

More generally, for $Z \subseteq Y \subseteq Z$, we have a fiber sequence

$$L(Y, Z; \mathcal{C}, Q) \to L(X, Z; \mathcal{C}, Q) \to L(X, Y; \mathcal{C}, Q).$$

Note that the ∞-category $\mathrm{Shv}_{\mathrm{const}}(X, Y; \mathcal{C})$ can be identified with the full subcategory of $\mathrm{Shv}_{\mathrm{const}}(X/Y; \mathcal{C})$ spanned by those sheaves which vanish at the base point of X/Y. For every pointed polyhedron Z, let $L^{\mathrm{red}}(Z; \mathcal{C}, Q)$ denote the relative L-theory space $L(Z, *; \mathcal{C}, Q)$. The construction $Z \mapsto L^{\mathrm{red}}(Z; \mathcal{C}, Q)$ is functorial with respect to pointed PL maps between pointed finite polyhedra. Moreover, Proposition 4 implies that it carries homotopy equivalences to homotopy equivalences, and therefore extends (in an essentially unique way) to a map

$$L^{\mathrm{red}}(\bullet; \mathcal{C}, Q) : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S},$$

where $\mathcal{S}_*^{\mathrm{fin}}$ denotes the ∞-category of pointed finite spaces.

Proposition 5. *The functor $L^{red}(\bullet; \mathcal{C}, Q) : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$ is excisive: that is, it carries homotopy pushout squares to homotopy pullback squares.*

Proof. Consider a homotopy pushout square of finite pointed spaces

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y'. \end{array}$$

Without loss of generality, we may assume that each of these spaces is a finite polyhedron, each of the maps are PL, the horizontal maps are inclusions. Consider the diagram

$$\begin{array}{ccccc} L^{\mathrm{red}}(X; \mathcal{C}, Q) & \longrightarrow & L^{\mathrm{red}}(X'; \mathcal{C}, Q) & \longrightarrow & L^{\mathrm{red}}(X'/X; \mathcal{C}, Q) \\ \downarrow & & \downarrow & & \downarrow \theta \\ L^{\mathrm{red}}(Y; \mathcal{C}, Q) & \longrightarrow & L^{\mathrm{red}}(Y'; \mathcal{C}, Q) & \longrightarrow & L^{\mathrm{red}}(Y'/Y; \mathcal{C}, Q). \end{array}$$

Since the rows are fiber sequences, to show that the left square is a homotopy pullback, it will suffice to show that θ is a homotopy equivalence. This is clear, since the map $X'/X \to Y'/Y$ is a homotopy equivalence, by virtue of our assumption that σ is a homotopy pushout square. $\square$

It follows from Proposition 5 that we can write

$$L^{\mathrm{red}}(X; \mathcal{C}, Q) \simeq \Omega^{\infty}(X \wedge \mathbf{L}(\mathcal{C}, Q))$$

for some spectrum $\mathbb{L}(\mathcal{C}, Q)$, which we will call the *L-theory spectrum* of the pair $(\mathcal{C}, Q)$. In particular, $L(X; \mathcal{C}, Q) \simeq L^{\mathrm{red}}(X_+; \mathcal{C}, Q)$ can be identified with the zeroth space of $X_+ \wedge \mathbb{L}(\mathcal{C}, Q)$. Taking X to be a point, we get $\Omega^{\infty}\mathbb{L}(\mathcal{C}, Q) = L(\mathcal{C}, Q)$, so that the homotopy groups of the spectrum $\mathbb{L}(\mathcal{C}, Q)$ are the L-groups of the pair $(\mathcal{C}, Q)$. More generally,

$$L_n(X; \mathcal{C}, Q) \simeq \pi_n(X_+ \wedge \mathbb{L}(\mathcal{C}, Q))$$

is the nth homology group of X with coefficients in the spectrum $\mathbb{L}(\mathcal{C}, Q)$.