Categorical Background (Lecture 2)

February 2, 2011

In the last lecture, we stated the main theorem of simply-connected surgery (at least for manifolds of dimension $4m$), which highlights the importance of the *signature* σ_X as an invariant of an (oriented) Poincare complex. Let us begin with a few remarks about how this invariant is defined.

Let V be a finite dimensional vector space over the real numbers and let $q : V \to \mathbb{R}$ be a nondegenerate quadratic form on V, with associated bilinear form $(,) : V \times V \to \mathbb{R}$. We can always choose an orthogonal basis $\{x_1, \ldots, x_a, y_1, \ldots, y_b\}$ for V satisfying

$$(x_i, x_i) = 1 \qquad (y_i, y_i) = -1.$$

The sum $a + b$ is the *dimension* of the vector space V, and the difference $a - b$ is called the *signature* of q, and denoted $\sigma(q)$.

Theorem 1 (Sylvester)**.** *Let V be a finite dimensional vector space over $\mathbb{R}$ equipped with a nondegenerate quadratic form $q : V \to \mathbb{R}$. Then the signature $\sigma(q)$ is well-defined: that is, it does not depend on the choice of orthogonal basis for V. Moreover, if V' is another finite dimensional vector space over $\mathbb{R}$ with a quadratic form $q' : V' \to \mathbb{R}$, then there exists an isometry $(V, q) \simeq (V', q')$ if and only if $\dim V = \dim V'$ and $\sigma(q) = \sigma(q')$.*

The notion of a vector space V with a quadratic form makes sense over an arbitrary field k. We have emphasized the case $k = \mathbb{R}$ for two reasons: first, the classification of quadratic forms over $\mathbb{R}$ is particularly simple (because of Theorem 1). Second, information about the intersection form on the middle cohomology $\mathrm{H}^{2m}(X; \mathbb{R})$ of a simply connected Poincare complex of dimension $4m$ plays an important role in determining whether or not X is homotopy equivalent to a manifold. However, quadratic forms over other fields are also of geometric interest. For example, if X is a simply connected Poincare complex of dimension $4m + 2 > 4$, then the problem of finding a manifold in the homotopy type of X turns out to depend on more subtle properties of quadratic forms over the field $\mathbf{F}_2$. We would ultimately like to treat manifolds of all dimensions in a uniform way, which will require us to somehow interpolate between the fields $k = \mathbb{R}$ and $k = \mathbf{F}_2$. We can do this by considering quadratic forms over the integers, or over more general rings.

The theory of quadratic forms over an arbitrary ring R can be very complicated. For example, the classification of quadratic forms over the field $\mathbf{Q}$ of rational numbers is a nontrivial achievement in number theory (the Hasse-Minkowski theorem). In general, we cannot detect whether two quadratic forms are isomorphic by means of simple integer invariants, as in Theorem 1. However, there are more elaborate theories that are designed to generalize the dimension and signature to other contexts:

	K-theory	L-theory
input:	projective module	module with quadratic form
for $\mathbb{R}$-vector spaces	dimension	signature
classical version	Grothendieck group K_0	Witt group
invariant of manifolds	Euler characteristic	signature
local-global principle	Gauss-Bonnet theorem	Hirzebruch signature formula

We are ultimately interested in understanding the right hand column of this table. But let us first spend some time discussing the left column, which is perhaps more familiar. We begin by recalling the definition of the Grothendieck group $K_0(A)$ of an associative ring A. The collection of isomorphism classes of finitely generated projective A-modules forms a commutative monoid under the formation of direct sums. The group completion of this monoid is denoted by $K_0(A)$, and called the (0th) K-group of A. Put another way, the K-group $K_0(A)$ is the abelian group generated by symbols $[P]$, where P ranges over all projective left A-modules of finite rank, subject to the relations given by

$$[P \oplus P'] = [P] + [P'].$$

Remark 2. If A is a field, then all finitely generated A-modules are automatically projective, and are determined up to isomorphism by their rank (in other words, by their dimension as A-vector spaces). Consequently, there is a canonical isomorphism $K_0(A) \simeq \mathbf{Z}$, which assigns to each A-module P its dimension $\dim_A(P)$. Consequently, we can think of the K-group $K_0(A)$ as a device which allows us to generalize the notion of dimension to the case of modules over arbitrary rings.

Let us now consider the following question:

Question 3. What is K-theory an invariant of?

We give several answers, beginning with the obvious.

(a) K-theory is an invariant of rings.

However, we can immediately improve on (a). Note that the Grothendieck group $K_0(A)$ is defined purely in terms of the category of finitely generated projective modules over A. In particular, Morita equivalent rings (such as matrix rings $M_n(A)$) have the same K-theory as A. We can therefore improve on our first answer:

(b) K-theory is an invariant of additive categories.

For our purposes, it will be convenient to give a variation on this answer. Recall that we are ultimately interested in assigning geometric invariants to manifolds and other Poincare complexes. We can obtain some by considering algebraic invariants to vector spaces. Let k be a field. For any finite CW complex X, we can consider the *Betti numbers*

$$b_i = \dim_k \mathrm{H}_i(X; k).$$

In general, these invariants depend on the field k. However, the Euler characteristic

$$\chi(X) = \sum_i (-1)^i b_i = \sum_i (-1)^i \dim_k \mathrm{H}_i(X; k)$$

does not depend on k. This invariant $\chi(X)$ is not generally not the dimension of a vector space (for example, it can be negative). Instead, we should think of it as an invariant of a *chain complex* of vector spaces: the singular chain complex $C_*(X;k)$.

More generally, we would like to say that for any ring A, the chain complex $C_*(X;A)$ determines a class in the Grothendieck group $K_0(A)$. (Of course, we know what this class should be: namely, $\chi(X)[A] \in K_0(A)$. Ignore this for the moment.) We generally cannot define this class to be the alternating sum

$$\sum_i (-1)^i [\mathrm{H}_i(X;A)],$$

because the individual A-modules $\mathrm{H}_i(X;A)$ need not be projective.

To show that $C_*(X;A)$ determines a well-defined class in the group $K_0(A)$, it is convenient to describe $K_0(A)$ in a different way. Rather than representing K-theory classes by projective modules over A, we take as representatives *chain complexes* of modules over A. Of course, we do not want to allow arbitrary chain complexes. In order to obtain reasonable invariants, we should restrict our attention to chain complexes which are *perfect*: that is, which are quasi-isomorphic to bounded chain complexes of finite projective modules. If X is a finite CW complex, then the singular chain complex $C_*(X;A)$ is always perfect: for example, it is quasi-isomorphic to the chain complex which computes the cellular homology of X.

Let **Perf** denote the category whose objects are bounded chain complexes of finite projective modules. For every perfect complex $M_\bullet$, we can find a quasi-isomorphism $P_\bullet \to M_\bullet$, where $P_\bullet \in \mathbf{Perf}$. The chain complex $P_\bullet$ need not be unique. However, it is unique up to chain homotopy equivalence. We can obtain a stronger uniqueness result by passing to the homotopy category. Let **hPerf** be the category with the same objects as **Perf**, but whose morphisms are given by *chain homotopy classes* of chain maps. The category **hPerf** is called the *perfect derived category of A*. Every perfect complex of A-modules determines an object of **hPerf**, which is well-defined up to isomorphism. Moreover, **hPerf** is an example of a *triangulated category*: that is, there is a notion of *distinguished triangle*

$$P'_\bullet \to P_\bullet \to P''_\bullet \to P'_\bullet[1]$$

in **hPerf**. Let $K'_0(A)$ denote the abelian group generated by symbols $[P_\bullet]$, where $P_\bullet$ is an object of **hPerf**, with relations $[P_\bullet] = [P'_\bullet] + [P''_\bullet]$ for every distinguished triangle

$$P'_\bullet \to P_\bullet \to P''_\bullet \to P'_\bullet[1].$$

Every finitely generated projective A-module can be regarded as a perfect complex which is concentrated in degree zero, and this construction determines a map of abelian groups $K_0(A) \to K'_0(A)$. One can show that this map is an isomorphism. In other words, one can define the Grothendieck group using (perfect) chain complexes of modules, rather than individual modules. With this definition, it is easy to see that $C_*(X;A)$ determines a well-defined K-theory class, for every finite CW complex X. This suggests a different answer to Question 3:

(c) K-theory is an invariant of triangulated categories.

For purposes of this course, answer (c) is a little bit misleading. The passage from the category **Perf** to its homotopy category **hPerf** has upsides and downsides. It has the virtue of allowing us to treat quasi-isomorphic chain complexes as if they are actually isomorphic. Unfortunately, it also loses a lot of information, because it has the effect of identifying chain homotopic morphisms without remembering any information about the chain homotopy. For our purposes, we will need to work with an intermediate object $\mathcal{D}^{\mathrm{perf}}(A)$, which will allow us to treat quasi-isomorphisms between chain complexes as if they were isomorphisms while retaining information about chain homotopies. The fine print is that unlike **Perf** and **hPerf**, $\mathcal{D}^{\mathrm{perf}}(A)$ is not a category: rather it is a more general object called an ∞-category.

We are now ready to give our final answer to Question 3:

(d) K-theory is an invariant of stable ∞-categories.

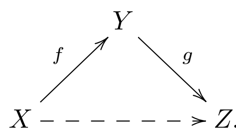
Our goal in this lecture and the next is to explain the meaning of this statement. To do so will require a long digression.

Definition 4. Let $\mathcal{C}$ be a category. The *nerve* of $\mathcal{C}$ is a simplicial set $\mathrm{N}(\mathcal{C})$ whose n-simplices are given by composable sequences of morphisms

$$C_0 \to C_1 \to \cdots \to C_n$$

in $\mathcal{C}$.

If $\mathcal{C}$ is a category, then $\mathcal{C}$ can be recovered (up to canonical isomorphism) from its nerve $\mathrm{N}(\mathcal{C})$. The objects of $\mathcal{C}$ are in bijection with the 0-simplices of $\mathrm{N}(\mathcal{C})$. If X and Y are objects in $\mathcal{C}$, then $\mathrm{Map}_{\mathcal{C}}(X, Y)$ can be identified with the set of 1-simplices of $\mathrm{N}(\mathcal{C})$ joining X with Y. Given a pair of composable morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, there is a unique 2-simplex of $\mathrm{N}(\mathcal{C})$ having 0th face g and 2nd face f, and the composition $g \circ f$ is the 1st face of this 2-simplex:



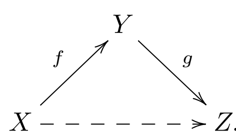
We can summarize this discussion as follows: the construction $\mathcal{C} \mapsto \mathrm{N}(\mathcal{C})$ determines a fully faithful embedding from the category of (small) categories into the category of simplicial sets. The essential image is described by the following claim:

Fact 5. *Let S be a simplicial set. Then S is isomorphic to the nerve of a category if and only if the following condition is satisfied:*

- $(*)$ *For every pair of integers $0 < i < n$, every map $f_0 : \Lambda^n_i \to S$ extends uniquely to an n-simplex $f : \Delta^n \to S$.*

Here Λ^n_i denotes the ith horn: the simplicial subset of Δ^n obtained by removing the interior and the face opposite the ith vertex.

Example 6. When $i = 1$ and $n = 2$, condition $(*)$ says that every pair of "composable" edges f and g determine a unique 2-simplex



Recall that a simplicial set S is a *Kan complex* if it satisfies the following variant of $(*)$:

- $(*')$ For every pair of integers $0 \le i \le n$, every map $f_0 : \Lambda^n_i \to S$ extends to an n-simplex $f : \Delta^n \to S$.

Conditions $(*)$ and $(*')$ look similar, but neither implies the other. Condition $(*)$ requires that we can uniquely fill any *inner* horn (that is, a horn Λ^n_i with $0 < i < n$), but says nothing about the extremal cases $i = 0$ and $i = n$. Condition $(*')$ requires that we can fill every horn Λ^n_i, but does not require the filler to be unique. However, these two conditions admit a common generalization:

Definition 7. An ∞*-category* is a simplicial set S satisfying the following condition:

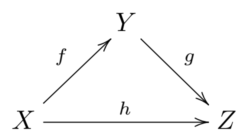
- $(*'')$ For every pair of integers $0 < i < n$, every map $f_0 : \Lambda^n_i \to S$ extends to an n-simplex $f : \Delta^n \to S$.

Remark 8. In the literature, ∞-categories are often referred to as *quasi-categories* or *weak Kan complexes.*

Example 9. If $\mathcal{C}$ is a category, then its nerve $\mathrm{N}(\mathcal{C})$ is an ∞-category. Since passage to the nerve loses no information about a category, this construction allows us to view the usual definition of a category as a special case of the notion of ∞-category.

Example 10. Any Kan complex is an ∞-category. In particular, if X is a topological space, then the singular complex $\mathrm{Sing}(X)$ (whose n-simplices are given by continuous maps from a topological n-simplex into X) is an ∞-category. Since the singular complex $\mathrm{Sing}(X)$ determines X up to weak homotopy equivalence, not much information is lost by the construction $X \mapsto \mathrm{Sing}(X)$. Consequently, for many purposes, we can think of ∞-categories as a generalization of topological spaces.

We will typically use the symbol $\mathcal{C}$ to denote an ∞-category. We will refer to the 0-simplices of $\mathcal{C}$ as its *objects* and the 1-simplices of $\mathcal{C}$ as its *morphisms*. In the simplest case ($i = 1$ and $n = 2$), the horn-filling condition $(*'')$ asserts that for every pair of "composable" morphisms $f : X \to Y$ and $g : Y \to Z$, we can find a 2-simplex σ :



in $\mathcal{C}$. Here we can think of h as a *composition* of f and g, and we will write $h = g \circ f$. A word of warning is in order: condition $(*'')$ does not require that σ is unique, so there may be several choices for the composition h. However, one can show that h is unique up to a suitable notion of homotopy. This turns out to be good enough for many purposes: Definition 7 provides a robust generalization of classical category theory. Many of the useful concepts from classical category theory (limits and colimits, adjoint functors, etcetera) can be generalized to the setting of ∞-categories.