Verdier Duality (Lecture 19)

March 9, 2011

Fix a polyhedron X and a stable ∞-category $\mathcal{C}$.

Construction 1. Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a quadratic functor. Let T be a triangulation of X. We define $Q_T : \mathrm{Shv}^c_T(\mathcal{C})^{op} \to \mathrm{Sp}$ by the formula

$$Q_T(\mathcal{F}) = \varinjlim_{\tau \in T} Q(\mathcal{F}(\tau)).$$

It is not difficult to see that Q_T is a quadratic functor on $\mathrm{Shv}^c_T(K, \mathcal{C})^{op}$, whose associative bilinear functor is given by

$$B_T(\mathcal{F}, \mathcal{F}') = \varinjlim_{\tau \in T} B(\mathcal{F}(\tau), \mathcal{F}'(\tau)),$$

where B is the bilinear functor associated to Q.

In the situation of Construction 1, suppose we are given another triangulation S of X which refines T. Let $i^* : \mathrm{Shv}^c_T(X; \mathcal{C}) \to \mathrm{Shv}^c_S(X; \mathcal{C})$ be as defined earlier. The composition

$$\mathrm{Shv}^c_T(X; \mathcal{C})^{op} \xrightarrow{i^*} \mathrm{Shv}^c_S(X; \mathcal{C}) \xrightarrow{Q_S}$$

is given by the formula $\mathcal{F} \mapsto \varinjlim_{\sigma \in S} Q(\mathcal{F}(i(\sigma)))$. We claim that this composition is canonically equivalent to Q_T. To prove this, it suffices to show that i induces a cofinal map $S^{op} \to T^{op}$. Unwinding the definitions, we must show that for every simplex $\tau \in T$, the partially ordered set $\{\sigma \in S : i(\sigma) \subseteq \tau\}$ has weakly contractible nerve. This is clear, since the geometric realization of this nerve is homeomorphic to the simplex τ. It follows that the functors Q_T are compatible as T ranges over all triangulations of X, and amalgamate to a quadratic functor

$$Q_{\mathrm{const}} : \mathrm{Shv}^c_{\mathrm{const}}(X; \mathcal{C})^{op} \to \mathrm{Sp}\,.$$

Let us now assume that the quadratic functor Q on $\mathcal{C}$ is representable, and let $\mathbb{D}$ be the corresponding duality functor. We claim that for every triangulation T of K, the quadratic functor Q_T is also representable, and the corresponding duality functor $\mathbb{D}_T$ is given by the formula

$$\mathbb{D}_T(\mathcal{F})(\tau) = \varinjlim_{\sigma} \begin{cases} \mathbb{D}(\mathcal{F}(\sigma)) & \text{if } \tau \subseteq \sigma \\ 0 & \text{otherwise.} \end{cases}$$

(Note that this functor carries $\mathrm{Shv}^c_T(K; \mathcal{C})$ to itself).

For simplicity, let us assume that $\mathbb{D}_T$ exists and show that it is given by the formula above. (A slightly more complication version of the same argument will show that it is given by the above formula.) Fix an object $C \in \mathcal{C}$ and a simplex $\tau \in T$. For every object $\mathcal{F} \in \mathrm{Shv}_T(X; \mathcal{C})$, we have homotopy equivalences

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{C}}(C, (\mathbb{D}_T \mathcal{F})(\tau)) &\simeq \mathrm{Mor}_{\mathrm{Shv}_T(X;\mathcal{C})}(\mathcal{F}^{\tau,C}, \mathbb{D}_T \mathcal{F}) \\
&\simeq B_T(\mathcal{F}^{\tau,C}, \mathcal{F}) \\
&\simeq \varinjlim_{\tau' \in T} B(\mathcal{F}^{\tau,C}(\tau'), \mathcal{F}(\tau')) \\
&\simeq \varinjlim_{\tau'} \mathrm{Mor}_{\mathcal{C}}(C, \begin{cases} \mathbb{D}\,\mathcal{F}(\tau') & \text{if } \tau \subseteq \tau' \\ 0 & \text{otherwise.} \end{cases}) \\
&\simeq \mathrm{Mor}_{\mathcal{C}}(C, \varinjlim \begin{cases} \mathbb{D}\,\mathcal{F}(\tau') & \text{if } \tau \subseteq \tau' \\ 0 & \text{otherwise.} \end{cases})
\end{aligned}$$

depending functorially on C, so by Yoneda's lemma we get a canonical equivalence

$$(\mathbb{D}_T \mathcal{F})(\tau) = \varinjlim_{\tau'} \begin{cases} \mathbb{D}(\mathcal{F}(\tau)) & \text{if } \tau \subseteq \tau' \\ 0 & \text{otherwise.} \end{cases}$$

Example 2. Let $\mathcal{F}$ be a constant functor taking the value $C \in \mathcal{C}$.The above formula shows that

$$(\mathbb{D}_T \mathcal{F})(\tau) \simeq \mathbb{D}\Gamma(\mathcal{F}^{\tau,C}) = \mathbb{D}(C^{(X, X-\{x\})}) = \mathbb{D}(C) \wedge (X/X - \{x\})$$

(here $x \in X$ denotes a point belonging to the interior of σ, and we regard the stable ∞-category $\mathcal{C}$ as tensored over the ∞-category of finite pointed spaces).

Now suppose that S is a triangulation refining T. We claim that the functor

$$i^* : \mathrm{Shv}^c_T(X; \mathcal{C}) \to \mathrm{Shv}^c_{T'}(X; \mathcal{C})$$

intertwines the duality functors $\mathbb{D}_T$ and $\mathbb{D}_{T'}$. Since $Q_T \simeq Q_{T'} \circ i^*$, for every object $\mathcal{F} \in \mathrm{Shv}^c_T(X; \mathcal{C})$ we have a canonical map $i^*\mathbb{D}_T(\mathcal{F}) \to \mathbb{D}_{T'}(i^* \mathcal{F})$. We claim that this map is invertible. To prove this, consider an arbitrary $\mathcal{G} \in \mathrm{Shv}^c_S(X; \mathcal{C})$. We wish to prove that the canonical map

$$B_T(i_+ \mathcal{G}, \mathcal{F}) \simeq \mathrm{Mor}_{\mathrm{Shv}^c_T}(i_+ \mathcal{G}, \mathbb{D}_T \mathcal{F})\, \mathrm{Mor}_{\mathrm{Shv}^c_S(X;\mathcal{C})}(\mathcal{G}, i^*\mathbb{D}_T \mathcal{F}) \to \mathrm{Mor}_{\mathrm{Shv}^c_S(X;\mathcal{C})}(\mathcal{G}, \mathbb{D}_S i^* \mathcal{F}) \simeq B_S(\mathcal{G}, i^* \mathcal{F})$$

is an equivalence. The left hand side is given by

$$\varinjlim_{\tau \in T} B(\varinjlim_{\sigma \in S, \sigma \subseteq \tau} \mathcal{G}(\sigma), \mathcal{F}(\tau)) \simeq \varinjlim_{\tau \in T} \varprojlim_{\sigma \in S, \sigma \subseteq \tau} B(\mathcal{G}(\sigma), \mathcal{F}(\tau)).$$

In the last lecture, we saw that the collection of $\sigma \subseteq \tau$ such that $i(\sigma) = \tau$ is cofinal in the collection of all $\sigma \subseteq \tau$. We may therefore rewrite the above limit as

$$\varinjlim_{\tau \in T} \varprojlim_{\sigma \in S, \sigma \subseteq \tau} B(\mathcal{G}(\sigma), \mathcal{F}(i(\sigma)).$$

Define $\mathcal{H} : S \to \mathrm{Sp}^{op}$ by the formula $\mathcal{H}(\sigma) = B(\mathcal{G}(\sigma), \mathcal{F}(i\sigma))$. Then the above construction is given by $\Gamma(i_+ \mathcal{H})$ (computed in the ∞-category Sp^{op}). In the last lecture, we saw that this is equivalent to

$$\Gamma(\mathcal{H}) = \varinjlim_{\sigma \in S} B(\mathcal{G}(\sigma), \mathcal{F}(i(\sigma))) = B_S(\mathcal{G}, i^* \mathcal{F}),$$

as desired.

Amalgamating the duality functors $\mathbb{D}_T$ as T runs over all triangulations of X, we obtain a duality functor

$$\mathbb{VD} : \mathrm{Shv}^c_{\mathrm{const}}(X; \mathcal{C})^{op} \to \mathrm{Shv}^c_{\mathrm{const}}(X; \mathcal{C}).$$

We will refer to this functor as *Verdier duality.*

Suppose now that we have two finite polyhedra X and Y, and a PL map $f : X \to Y$. To f we can associate a pullback functor $f^* : \mathrm{Shv}_{\mathrm{const}}(Y; \mathcal{C}) \to \mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$. This pullback functor admits right adjoint, which we will denote by f_*. For $\mathcal{F} \in \mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})$, can explicitly describe $f_*(\mathcal{F})$ as follows. Choose triangulations S and T of X and Y, respectively, such that $\mathcal{F}$ is S-constructible and f is *simplicial*: that is, it induces a linear map from each simplex of S to each simplex of T (carrying vertices to vertices). Then $f_* \mathcal{F}$ can be identified with the T-constructible sheaf on Y given by the formula

$$(f_* \mathcal{F})(\tau) = \varprojlim_{\sigma \in S, \tau \subseteq f(\sigma)} \mathcal{F}(\sigma).$$

Using a cofinality argument, we can also write

$$(f_* \mathcal{F})(\tau) = \varprojlim_{f(\sigma) = \tau} \mathcal{F}(\sigma).$$

Let $\mathcal{G} \in \mathrm{Shv}_T(Y, \mathcal{C})$. Then we have a canonical equivalence

$$B_S(\mathcal{F}, f^* \mathcal{G}) = \varinjlim_{\sigma} B(\mathcal{F}(\sigma), \mathcal{G}(f(\sigma))) = \varinjlim_{\tau} \varinjlim_{f(\sigma)=\tau} B(\mathcal{F}(\sigma), \mathcal{G}(\tau)) \simeq \varinjlim_{\tau} (\varprojlim_{f(\sigma)=\tau} \mathcal{F}(\sigma), \mathcal{G}(\tau)) = B_T(f_* \mathcal{F}, \mathcal{G}).$$

We can rewrite the left side as

$$\mathrm{Mor}_{\mathrm{Shv}_S(X;\mathcal{C})}(f^* \mathcal{G}, \mathbb{D}_S \mathcal{F}) \simeq \mathrm{Mor}_{\mathrm{Shv}_T(Y;\mathcal{C})}(\mathcal{G}, f_* \mathbb{D}_S \mathcal{F})$$

and the right side as

$$\mathrm{Mor}_{\mathrm{Shv}_T(Y;\mathcal{C})}(\mathcal{G}, \mathbb{D}_T f_* \mathcal{F}).$$

Yoneda's lemma gives us a canonical isomorphism

$$\mathbb{D}_T f_* \mathcal{F} \simeq f_* \mathbb{D}_S \mathcal{F}.$$

Passing to the direct limit over all triangulations, we deduce that Verdier duality commutes with pushforwards.

Proposition 3. *Let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be a nondegenerate quadratic functor. Then for any finite polyhedron X equipped with a triangulation T, the quadratic functor $Q_T : \mathrm{Shv}_T(X; \mathcal{C})^{op} \to \mathrm{Sp}$ is nondegenerate. Passing to the direct limit over T, obtain a nondegenerate quadratic functor $Q_{\mathrm{const}} : \mathrm{Shv}_{\mathrm{const}}(X; \mathcal{C})^{op} \to \mathrm{Sp}$.*

Proof. Let $\mathcal{F} \in \mathrm{Shv}_T(X; \mathcal{C})$; we wish to show that the canonical map

$$\theta_{\mathcal{F}} : \mathcal{F} \to \mathbb{D}_T \mathbb{D}_T \mathcal{F}$$

is invertible. For every simplex $\tau \in T$ and every object $C \in \mathcal{C}$, define $\mathcal{F}_{\tau,C}$ by the formula

$$\mathcal{F}_{\tau,C}(\tau') = \begin{cases} C & \text{if } \tau' \subseteq \tau \\ 0 & \text{otherwise.} \end{cases}$$

Arguing as in the previous lecture, we see that the objects $\mathcal{F}_{\tau,C}$ generate the stable ∞-category $\mathrm{Shv}_T(X; \mathcal{C})$. It will therefore suffice to prove that $\theta_{\mathcal{F}}$ is an equivalence when $\mathcal{F} = \mathcal{F}_{\tau,C}$. Note that $\mathcal{F}_{\tau,C} = f_* \mathcal{G}$, where f denotes the inclusion $\tau \to X$ and $\mathcal{G}$ is a constant functor taking the value C. Since Verdier duality commutes with pushforwards, we can replace X by τ and thereby reduce to the case where X is a simplex, T is a the collection of faces of X, and $\mathcal{F}$ is the constant functor taking the value $C \in \mathcal{C}$. Let n be the dimension of X.

We now compute

$$\mathbb{D}_T(\mathcal{F})(\sigma) = \mathbb{D}(C^{(X, X - \{x\})})$$

where x is a point belonging to the interior of σ. If σ is a proper face of X, then X and $X - \{x\}$ are both contractible so that $\mathbb{D}_T(\mathcal{F})(\sigma) = 0$. If $\sigma = X$, then the cofiber $X/(X - \{x\})$ is homotopy equivalent to S^n, so that $\mathbb{D}_T(\mathcal{F})(\sigma) \simeq \Sigma^n \mathbb{D}(C)$. Thus $\mathbb{D}_T(\mathcal{F})$ is the functor $\mathcal{F}'$ given by the formula

$$\mathcal{F}'(\sigma) = \begin{cases} \Sigma^n \mathbb{D}(C) & \text{if } \sigma = X \\ 0 & \text{otherwise.} \end{cases}$$

We now compute $\mathbb{D}_T \mathcal{F}'$ using the formula

$$(\mathbb{D}_T \mathcal{F}')(\tau) = \varinjlim_{\sigma} \begin{cases} \mathbb{D} \mathcal{F}'(\sigma) & \text{if } \tau \subseteq \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Using our formula for $\mathcal{F}'$, we can rewrite this as

$$\varinjlim_{\sigma} \begin{cases} \mathbb{D}(\Sigma^n \mathbb{D}(C)) & \text{if } \sigma = X \\ 0 & \text{otherwise.} \end{cases}$$

The nondegeneracy of Q gives an equivalence $\mathbb{D}\Sigma^n \mathbb{D}(C) \simeq \Sigma^{-n} C$. We therefore obtain $(\mathbb{D}_T \mathcal{F}') \simeq (X, \partial X) \wedge \Sigma^{-n} C \simeq C$ for every simplex $\tau \in T$, so that we have an isomorphism $\mathbb{D}_T \mathcal{F}' \simeq \mathcal{F}$. With a bit more effort, one can show that this isomorphism is given by the functor $\theta_{\mathcal{F}}$ defined above. $\square$