

# Polyhedra and PL Manifolds (Lecture 17)

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In this lecture, we will review the notion of a *piecewise linear manifold* (which we will typically abbreviate as *PL manifold*). More information can be found in the lecture notes of my MIT course 18.937.

**Definition 1.** Let  $K$  be a subset of a Euclidean space  $\mathbb{R}^n$ . We will say that  $K$  is a *linear simplex* if it can be written as the convex hull of a finite subset  $\{x_1, \dots, x_k\} \subset \mathbb{R}^n$  which are independent in the sense that if  $\sum c_i x_i = 0 \in \mathbb{R}^n$  and  $\sum c_i = 0 \in \mathbb{R}$  imply that each  $c_i$  vanishes.

We will say that  $K$  is a *polyhedron* if, for every point  $x \in K$ , there exists a finite number of linear simplices  $\sigma_i \subseteq K$  such that the union  $\bigcup_i \sigma_i$  contains a neighborhood of  $X$ .

**Remark 2.** Any open subset of a polyhedron in  $\mathbb{R}^n$  is again a polyhedron.

**Remark 3.** Every polyhedron  $K \subseteq \mathbb{R}^n$  admits a *triangulation*: that is, we can find a collection of linear simplices  $S = \{\sigma_i \subseteq K\}$  with the following properties:

- (1) Any face of a simplex belonging to  $S$  also belongs to  $S$ .
- (2) Any nonempty intersection of any two simplices of  $S$  is a face of each.
- (3) The union of the simplices  $\sigma_i$  is  $K$ .

**Definition 4.** Let  $K \subseteq \mathbb{R}^n$  be a polyhedron. We will say that a map  $f : K \rightarrow \mathbb{R}^m$  is *linear* if it is the restriction of an affine map from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . We will say that  $f$  is *piecewise linear* (PL) if there exists a triangulation  $\{\sigma_i \subseteq K\}$  such that each of the restrictions  $f|_{\sigma_i}$  is linear.

If  $K \subseteq \mathbb{R}^n$  and  $L \subseteq \mathbb{R}^m$  are polyhedra, we say that a map  $f : K \rightarrow L$  is piecewise linear if the underlying map  $f : K \rightarrow \mathbb{R}^m$  is piecewise linear.

**Remark 5.** Let  $f : K \rightarrow L$  be a piecewise linear homeomorphism between polyhedra. Then the inverse map  $f^{-1} : L \rightarrow K$  is again piecewise linear. To see this, choose any triangulation of  $K$  such that the restriction of  $f$  to each simplex of the triangulation is linear. Taking the image under  $f$ , we obtain a triangulation of  $L$  such that the restriction of  $f^{-1}$  to each simplex is linear.

**Remark 6.** The collection of all polyhedra can be organized into a category, where the morphisms are given by piecewise linear maps. This allows us to think about polyhedra *abstractly*, without reference to an embedding into a Euclidean space: a pair of polyhedra  $K \subseteq \mathbb{R}^n$  and  $L \subseteq \mathbb{R}^m$  can be isomorphic even if  $n \neq m$ .

**Remark 7.** Let  $K$  be a polyhedron. The following conditions are equivalent:

- (1) As a topological space,  $K$  is compact.
- (2)  $K$  admits a triangulation having finitely many simplices.
- (3) Every triangulation of  $K$  has only finitely many simplices.

If these conditions are satisfied, we say that  $K$  is a *finite polyhedron*.

**Definition 8.** Let  $M$  be a polyhedron. We will say that  $M$  is a *piecewise linear manifold* (of dimension  $n$ ) if, for every point  $x \in M$ , there exists an open neighborhood  $U \subseteq M$  containing  $x$  and a piecewise linear homeomorphism  $U \simeq \mathbb{R}^n$ .

**Remark 9.** If  $M$  is a PL manifold of dimension  $n$ , then the underlying topological space of  $M$  is an  $n$ -manifold. We can think of a PL manifold as a topological manifold equipped with some additional structure. There are many ways to describe this additional structure. For example, let  $\mathcal{O}_M$  denote the sheaf of continuous real-valued functions on  $M$ . A PL structure on  $M$  determines a subsheaf  $\mathcal{O}_M^{PL}$ , which assigns to each open set  $U \subseteq M$  the collection of *piecewise linear* continuous functions  $U \rightarrow \mathbb{R}$ . As a polyhedron,  $M$  is determined by its underlying topological space together with the sheaf  $\mathcal{O}_M^{PL}$ , up to PL homeomorphism.

Let  $K$  be a polyhedron containing a vertex  $x$ , and choose a triangulation of  $K$  containing  $x$  as a vertex of the triangulation. The *star* of  $x$  is the union of those simplices of the triangulation which contain  $x$ . The *link* of  $x$  consists of those simplices belonging to the star of  $x$  which do not contain  $x$ . We denote the link of  $x$  by  $\text{lk}(x)$ .

As a subset of  $K$ , the link  $\text{lk}(x)$  of  $x$  depends on the choice of triangulation of  $K$ . However, one can show that as an abstract polyhedron,  $\text{lk}(x)$  is independent of the triangulation up to piecewise linear homeomorphism. Moreover,  $\text{lk}(x)$  depends only on a neighborhood of  $x$  in  $K$ .

If  $K = \mathbb{R}^n$  and  $x \in K$  is the origin, then the link  $\text{lk}(x)$  can be identified with the sphere  $S^{n-1}$  (which can be regarded as a polyhedron via the realization  $S^{n-1} \simeq \partial \Delta^n$ ). It follows that if  $K$  is any piecewise linear  $n$ -manifold, then the link  $\text{lk}(x)$  is equivalent to  $S^{n-1}$  for every point  $x \in K$ . Conversely, if  $K$  is any polyhedron such that every link in  $K$  is an  $(n-1)$ -sphere, then  $K$  is a piecewise linear  $n$ -manifold. To see this, we observe that for each  $x \in K$ , if we choose a triangulation of  $K$  containing  $x$  as a vertex, then the star of  $x$  can be identified with the cone on  $\text{lk}(x)$ . If  $\text{lk}(x) \simeq S^{n-1}$ , then the star of  $x$  is a piecewise linear (closed) disk, so that  $x$  has a neighborhood which admits a piecewise linear homeomorphism to the open disk in  $\mathbb{R}^n$ .

This argument proves the following:

**Proposition 10.** *Let  $K$  be a polyhedron. The following conditions are equivalent:*

- (i) *For each  $x \in K$ , the link  $\text{lk}(x)$  is a piecewise linear  $(n-1)$ -sphere.*
- (ii)  *$K$  is a piecewise linear  $n$ -manifold.*

**Remark 11.** Very roughly speaking, we can think of a piecewise linear manifold  $M$  as a topological manifold equipped with a triangulation. However, this is not quite accurate, since a polyhedron does not come equipped with a particular triangulation. Instead, we should think of  $M$  as equipped with a distinguished class of triangulations, which is stable under passing to finer and finer subdivisions.

**Warning 12.** Let  $K$  be a polyhedron whose underlying topological space is an  $n$ -manifold. Then  $K$  need not be a piecewise linear  $n$ -manifold: it is generally not possible to choose local charts for  $K$  in a piecewise linear fashion.

To get a feel for the sort of problems which might arise, consider the criterion of Proposition 10. To prove that  $K$  is a piecewise linear  $n$ -manifold, we need to show that for each  $x \in K$ , the link  $\text{lk}(x)$  is a (piecewise-linear)  $(n-1)$ -sphere. Using the fact that  $K$  is a topological manifold, we deduce that  $H_*(K, K - \{x\}; \mathbf{Z})$  is isomorphic to  $\mathbf{Z}$  in degree  $n$  and zero elsewhere; this is equivalent to the assertion that  $\text{lk}(x)$  has the homology of an  $(n-1)$ -sphere. Of course, this does not imply that  $\text{lk}(x)$  is itself a sphere. A famous counterexample is due to Poincaré: if we let  $I$  denote the binary icosahedral group, regarded as a subgroup of  $\text{SU}(2) \simeq S^3$ , then the quotient  $P = \text{SU}(2)/I$  is a homology sphere which is not a sphere (since it is not simply connected).

The suspension  $\Sigma P$  is a 4-dimensional polyhedron whose link is isomorphic to  $P$  at precisely two points, which we will denote by  $x$  and  $y$ . However,  $\Sigma P$  is not a topological manifold. To see this, we note that the point  $x$  does not contain arbitrarily small neighborhoods  $U$  such that  $U - \{x\}$  is simply connected. In other words, the failure of  $\Sigma P$  to be a manifold can be detected by computing the *local fundamental group*

of  $P - \{x\}$  near  $x$  (which turns out to be isomorphic to the fundamental group of  $P$ ). However, if we apply the suspension functor again, the same considerations do not apply: the space  $\Sigma P$  is simply connected (by van Kampen's theorem). Surprisingly enough, it turns out to be a manifold:

**Theorem 13** (Cannon-Edwards). *Let  $P$  be a topological  $n$ -manifold which is a homology sphere. Then the double suspension  $\Sigma^2 P$  is homeomorphic to an  $(n + 2)$ -sphere.*

In particular, if we take  $P$  to be the Poincaré homology sphere, then there is a homeomorphism  $\Sigma^2 P \simeq S^5$ . However,  $\Sigma^2 P$  is not a piecewise linear manifold: it contains two points whose links are given by  $\Sigma P$ , which is not even a topological 4-manifold (let alone a piecewise linear 4-sphere).

The upshot of Warning 12 is that a topological manifold  $M$  (such as the 5-sphere) admits triangulations which are badly behaved, in the sense that the underlying polyhedron is not locally equivalent to Euclidean space.

Let us now review the relationship between smooth and PL manifolds.

**Definition 14.** Let  $K$  be a polyhedron and  $M$  a smooth manifold. We say that a map  $f : K \rightarrow M$  *piecewise differentiable* (PD) if there exists a triangulation of  $K$  such that the restriction of  $f$  to each simplex is smooth. We will say that  $f$  is a *PD homeomorphism* if  $f$  is piecewise differentiable, a homeomorphism, and the restriction of  $f$  to each simplex has injective differential at each point. In this case, we say that  $f$  is a *Whitehead triangulation* of  $M$ .

The problems of smoothing and triangulating manifold can be formulated as follows:

- (i) Given a smooth manifold  $M$ , does there exist a piecewise linear manifold  $N$  and a PD homeomorphism  $N \rightarrow M$ ?
- (ii) Given a piecewise linear manifold  $N$ , does there exist a smooth manifold  $M$  and a PD homeomorphism  $N \rightarrow M$ ?

Question (i) is addressed by the following theorem of Whitehead.

**Theorem 15** (Whitehead). *Let  $M$  be a smooth manifold. Then  $M$  admits a Whitehead triangulation. That is, there is exists a polyhedron  $K$  and a PD homeomorphism  $f : K \rightarrow M$ . Moreover,  $K$  is automatically a PL manifold, and is uniquely determined up to PL homeomorphism. (In fact, one can say more:  $K$  is uniquely determined up to a contractible space of choices. We will return to this point in a future lecture.)*

Problem (ii) is more subtle. In general, piecewise linear manifolds cannot be smoothed (Kervaire) and can admit inequivalent smoothings (Milnor's exotic spheres give examples of smooth manifolds which are not diffeomorphic, but whose underlying PL manifolds are PL homeomorphic). However, the difference between smooth and PL manifolds is governed by an *h-principle*. Given a PL manifold  $N$ , the problem of finding a PD homeomorphism to a smooth manifold can be rephrased as a homotopy lifting problem

$$\begin{array}{ccc}
 & & BO(n) \\
 & \nearrow & \downarrow \\
 N & \longrightarrow & BPL(n).
 \end{array}$$

(We will return to this point in more detail later, when we discuss microbundles.) In other words, the problem of finding a smooth structure on a PL manifold is equivalent to the problem of finding a suitable candidate for its tangent bundle.