The Even L-groups of $\mathbf{Z}$ (Lecture 16)

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In this lecture we will compute the quadratic L-groups of $\mathbf{Z}$ in even dimensions. We begin by considering $L^q_{-2}(\mathbf{Z})$. Consider the pair $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}, \Sigma^2 Q^q)$. Note that $\Sigma^2 Q^q(\Sigma \mathbf{Z}) \simeq \Sigma^2 (\Sigma^{-2}\mathbf{Z})_{h\Sigma_2}$ can be identified with the homotopy coinvariants of the group Σ_2 acting on the Eilenberg-MacLane spectrum corresponding to $\mathbf{Z}$, where the action is via the sign representation. In particular, we deduce that $\pi_0 \Sigma^2 Q^q(\Sigma \mathbf{Z})$ is isomorphic to the group $\mathbf{Z}/2\mathbf{Z}$.

Let (M, q) be a Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}, \Sigma^2 Q^q)$. Using surgery below the middle dimension, we can replace (M, q) by a cobordant Poincare object which is concentrated in degree 1: that is, we can assume that $M = \Sigma F$ for some finitely generated free abelian group L (which we identify with the corresponding Eilenberg-MacLane spectrum). Every element $\eta \in L$ determines a map $\Sigma \mathbf{Z} \to M$, so that $q | \Sigma \mathbf{Z}$ determines an element $e(\eta) \in \pi_0(\Sigma^2 Q^q(\Sigma \mathbf{Z})) \simeq \mathbf{Z}/2\mathbf{Z}$.

We can understand the construction $\eta \mapsto e(\eta)$ more explicitly as follows. Consider the surjective ring homomorphism $\mathbf{Z} \to \mathbf{F}_2$, where $\mathbf{F}_2$ denotes the finite field with two elements. Then $\mathbf{F}_2 \wedge_{\mathbf{Z}} M \simeq \Sigma(L/2L)$ inherits the structure of a Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{F}_2}, \Sigma^2 Q^q)$. Since $\mathbf{F}_2$ has characteristic 2, we can ignore signs and identify the reduction of q modulo 2 as a quadratic form q_0 on the $\mathbf{F}_2$-vector space $L/2L$. The invariant e is given by the composition

$$L \to L/2L \xrightarrow{q_0} \mathbf{Z}/2\mathbf{Z}.$$

Proposition 1. *The canonical map $L^q_{-2}(\mathbf{Z}) \to L^q_{-2}(\mathbf{F}_2) \simeq W(\mathbf{F}_2) \simeq \mathbf{Z}/2\mathbf{Z}$ is an isomorphism.*

Proof. We first prove injectivity. Let (M, q) be a Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}, \Sigma^2 Q^q)$, and assume as above that $M = \Sigma L$ for some finitely generated free abelian group L. Suppose that the image of (M, q) in $W(\mathbf{F}_2)$ is trivial; we wish to show that (M, q) is nullcobordant. We proceed by induction on the rank of L. If this rank is positive, then the quadratic form q_0 on $L/2L$ cannot be anisotropic (since $(M, q) \mapsto 0 \in W(\mathbf{F}_2)$). We can therefore choose a nonzero element $\overline{\eta} \in L/2L$ such that $q_0(\overline{\eta}) = 0$. Lift $\overline{\eta}$ to an element $\eta \in L$. Then η is not divisible by 2 (since $\overline{\eta} \neq 0$). Dividing η by an odd integer if necessary, we may assume that $\mathbf{Z}\eta$ is a direct summand of L. Since $e(\eta) = q_0(\overline{\eta}) = 0$, we can do surgery along η to obtain a new Poincare object (M', q'). Note that the homotopy groups of M' are given by the homology of the chain complex

$$\mathbf{Z}\eta \to L \xrightarrow{\phi} \mathbf{Z}.$$

The indivisibility of η (and nondegeneracy of q) imply that ϕ is surjective, so that the homology of this chain complex is concentrated in a single degree (and is therefore free, by duality). Moreover, the rank of the relevant homology group has dropped by 2, so we can finish using the inductive hypothesis.

It remains to show that the map $L^q_{-2}(\mathbf{Z}) \to W(\mathbf{F}_2)$ is surjective. For this, we just have to show that the nontrivial element of $W(\mathbf{F}_2)$ can be lifted to $L^q_{-2}(\mathbf{Z})$. We can represent this element by the nondegenerate quadratic space (V, q_0), where V is a two-dimensional vector space over $\mathbf{F}_2$ with basis $x, y \in V$, and q_0 is given by the formula $q_0(ax + a'y) = a^2 + aa' + a'^2$. Let $b_0 : V \times V \to \mathbf{F}_2$ be the bilinear form given by

$$b_0(x, x) = b_0(x, y) = b_0(y, y) = 1 \qquad b_0(y, x) = 0.$$

Then q_0 is the quadratic form associated to b_0. Note that b_0 lifts to a bilinear form b on $\mathbf{Z}x \oplus \mathbf{Z}y$, given by

$$b(x,x) = b(x,y) = b(y,y) = 1 \qquad b(y,x) = 0.$$

The associated skew-symmetric bilinear form

$$\epsilon(v,w) = b(v,w) - b(w,v)$$

is nondegenerate (since $\epsilon(x,y) = 1$). Note that b determines a point $q \in \Omega^\infty \Sigma^2 Q^q(\Sigma(\mathbf{Z}x \oplus \mathbf{Z}y))$, and that $(\mathbf{Z}x \oplus \mathbf{Z}y, q)$ is a Poincare object lifting (V, q_0). □

We now compute the group $L_0^q(\mathbf{Z})$. The inclusion $\mathbf{Z} \hookrightarrow \mathbb{R}$ determines a map $\psi : L_0^q(\mathbf{Z}) \to L_0^q(\mathbb{R}) = \mathbf{Z}$.

Proposition 2. *The map ψ is injective. Its image is the subgroup $8\mathbf{Z} \subseteq \mathbf{Z}$.*

Proof. By surgery below the middle dimension, we see that every element of $L_0^q(\mathbf{Z})$ can be represented by a pair (M, q), where M is a free abelian group of finite rank and q is a nondegenerate quadratic form on M. Here q is determined by its associated bilinear form $b : M \times M \to \mathbf{Z}$, given by $b(x,y) = q(x+y) - q(x) - q(y)$. This symmetric bilinear form is *even* (for any element $x \in M$ we have $b(x,x) = q(2x) - q(x) - q(x) = 2q(x)$) and *unimodular* (that is, it induces an isomorphism of abelian groups $M \to \mathrm{Hom}(M, \mathbf{Z})$). Conversely, any even symmetric bilinear form b determines a quadratic form q by the formula $q(x) = \frac{b(x,x)}{2}$, which is nondegenerate if and only if b is unimodular. We will prove the proposition by citing some nontrivial results about the structure of even unimodular lattices. Proofs can be found, for example, in Serre's book "A course in arithmetic."

The facts we need are the following:

- The image of ψ is contained in the subgroup $8\mathbf{Z} \subseteq \mathbf{Z}$. More concretely, we assert that if (M, q) is any even unimodular lattice, then the signature of M is divisible by 8.

- The map ψ is surjective: that is, there exists an even unimodular lattice of signature 8. In fact, there exists a *unique* positive definite even unimodular lattice of rank (and therefore signature) 8, the E_8-lattice. All we need here is the existence. For this, we can give a direct construction. Let L be the free abelian group on generators $e_1, \ldots, e_8$ and h. Equip it with an symmetric bilinear form b so that the generators are orthogonal and $b(e_i, e_i) = 1$, $b(h,h) = -1$. This is an odd unimodular lattice of signature 7. Let v denote the vector $e_1 + e_2 + \ldots + e_8 + 3h$. Note that $b(v,v) = 8(1) + 3^2(-1) = -1$. It follows that L splits as a direct sum $L_0 \oplus \mathbf{Z}v$, where L_0 is a unimodular lattice of rank 8 and signature 8. Note that for every $w \in L$, the integers $b(w,w)$ and $b(w,v)$ are congruent modulo 2 (it suffices to check this on generators, where it is obvious). Thus L_0 is an even unimodular lattice of signature 8.

- The map ψ is injective. Suppose we are given an even unimodular lattice (M, q) of signature zero. Let n be the rank of M; note that n must be even, since M is nondegenerate modulo 2. Let H denote the hyperbolic plane: that is, $(\mathbf{Z}^2, q_0)$ where q_0 is the quadratic form given by $q_0(a,b) = ab$. Then $H^{\oplus \frac{n}{2}}$ and (M, q) are indefinite even unimodular lattices of the same rank and signature. It follows from the theory of quadratic forms that $H^{\oplus \frac{n}{2}}$ and (M, q) are isomorphic. Consequently, to prove that (M, q) is nullcobordant, it suffices to show that H is nullcobordant, which is obvious.

□

Combining the above results with the 4-fold periodicity of $L_*^q(\mathbf{Z})$, we obtain the following:

Theorem 3. *The quadratic L-groups of $\mathbf{Z}$ are given by*

$$L_n^q(\mathbf{Z}) = \begin{cases} 8\mathbf{Z} & \text{if } n = 4k \text{ (signature)} \\ 0 & \text{if } n = 4k+1 \\ \mathbf{Z}/2\mathbf{Z} & \text{if } n = 4k+2 \text{ (Kervaire invariant)} \\ 0 & \text{if } n = 4k+3. \end{cases}$$

Let us conclude by saying a few words about the symmetric L-groups $L^s_*(\mathbf{Z})$. The norm map $Q^q \to Q^s$ induces a map

$$\phi : L^q_*(\mathbf{Z}) \to L^s_*(\mathbf{Z}).$$

Note that for any pair of spectra X and Y with actions of the group Σ_2, there is a canonical map

$$X_{h\Sigma_2} \wedge Y^{h\Sigma_2} \to (X \wedge Y)_{h\Sigma_2}.$$

Consequently, if (M, q) is a Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}, Q^s)$, then $M \otimes E_8$ inherits the structure of a Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}, Q^q)$, where E_8 denotes the E_8 lattice. This construction determines a map of L-groups

$$\psi : L^s_*(\mathbf{Z}) \to L^q_*(\mathbf{Z}).$$

The composite map $\phi \circ \psi : L^s_*(\mathbf{Z}) \to L^s_*(\mathbf{Z})$ is also given by tensoring with the E_8 lattice: this time, viewed as a lattice equipped with a symmetric bilinear form. Our construction of the E_8 lattice above shows that E_8 is stably isomorphic to the lattice $\mathbf{Z}^8$ with orthonormal basis $e_1, \ldots, e_8$: in fact, they become isomorphic after taking the direct sum with the unimodular lattice of rank 1 and signature -1. It follows that $\phi \circ \psi$ is given by multiplication by 8. In particular, ψ induces an injection $L^s_*(\mathbf{Z})[\frac{1}{2}] \to L^q_*(\mathbf{Z})[\frac{1}{2}]$ whose image is a summand of $L^q_*[\frac{1}{2}]$. This map is surjective in degree 0 (it hits the E_8 lattice by construction, which is a generator for $L^q_0(\mathbf{Z})$). By periodicity, it is surjective in degree $4k$ for every integer k. It is therefore surjective in all degrees (since $L^q_n(\mathbf{Z})[\frac{1}{2}] \simeq 0$ when n is not divisible by 4, by Theorem 3). It follows that ψ is an isomorphism after inverting 2. This proves:

Proposition 4. *The map ϕ induces an isomorphism $L^q_*(\mathbf{Z})[\frac{1}{2}] \to L^s_*(\mathbf{Z})[\frac{1}{2}]$. In particular, we have*

$$L^s_n(\mathbf{Z})[\frac{1}{2}] \simeq \begin{cases} \mathbf{Z}[\frac{1}{2}] & \text{if } n = 4k \\ 0 & \text{otherwise.} \end{cases}$$

With more effort, it is possible to compute the symmetric L-groups of $\mathbf{Z}$ precisely. The answer is given by

$$L^s_n(\mathbf{Z}) \simeq \begin{cases} \mathbf{Z} & \text{if } n = 4k \\ \mathbf{Z}/2\mathbf{Z} & \text{if } n = 4k+1 \\ 0 & \text{otherwise.} \end{cases}$$