Odd L-Theory of the Integers (Lecture 15)

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Our goal now is to compute the quadratic L-groups $L^q_n(\mathbf{Z})$. We have seen that the answers depend only on the congruence class of n modulo 4. We therefore have four calculations to perform. We will compute the odd L-groups of $\mathbf{Z}$ in this lecture, and the even L-groups of $\mathbf{Z}$ in the next lecture.

We begin with the easy part:

Proposition 1. *The quadratic groups $L^q_n(\mathbf{Z})$ vanish when n is congruent to 1 modulo 4.*

Proof. Write $n = -4k+1$. Let (M, q) be a quadratic object of $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}, \Sigma^{4k-1}Q^q)$. For every integer i, the homotopy group $\pi_i M$ is a finitely generated abelian group. We may therefore write $\pi_i M$ (noncanonically) as a direct sum $F_i \oplus T_i$, where F_i is a finitely generated free abelian group and T_i is torsion.

Let us identify abelian groups with the corresponding Eilenberg-MacLane spectra. Using the fact that the ring $\mathbf{Z}$ has projective dimension 1, we see that M is isomorphic to the direct sum

$$\bigoplus \Sigma^i F_i \oplus \Sigma^i T_i.$$

Moreover, this sum is finite (since M is perfect). The $\mathbf{Z}$-linear dual of M is given by

$$\bigoplus (\Sigma^{-i} \operatorname{Hom}(F_i, \mathbf{Z}) \oplus \Sigma^{-i-1} \operatorname{Ext}(T_i, \mathbf{Z})).$$

Since (M, q) is Poincare, we have an isomorphism $M \simeq \Sigma^{4k-1}\mathbb{D}M$. Passing to homotopy groups (and using the structure theory of finitely generated abelian groups) we obtain isomorphisms

$$F_i \simeq \operatorname{Hom}(F_{4k-1-i}, \mathbf{Z}) \qquad T_i \simeq \operatorname{Ext}(T_{4k-1-i}, \mathbf{Z}).$$

Replacing (M, q) by a cobordant Poincare object if necessary, we can assume that M is $(2k-1)$-connective: that is, the abelian groups F_i and T_i vanish for $i < 2k-1$. From duality, we deduce that $F_i \simeq 0$ for $i > 2k$ and that $T_i \simeq 0$ for $i \geq 2k$. Let $F = F_{2k-1}$ and $T = T_{2k-1}$, so that $\pi_{2k} M \simeq \operatorname{Hom}(F_k, \mathbf{Z})$ and $T \simeq \operatorname{Ext}(T, \mathbf{Z})$.

Let us now try to kill the torsion group T. Choose a nonzero element $\eta \in T$, classifying a map $\alpha : \Sigma^{2k-1}\mathbf{Z} \to M$. Recall that $\Sigma^{4k-1}Q^q(\Sigma^k \mathbf{Z})$ is a connected spectrum, so that $q|(\Sigma^k\mathbf{Z})$ automatically vanishes and we can therefore do surgery along α.

We have a fiber sequence of spectra

$$\Sigma^{2k-1}\mathbf{Z} \xrightarrow{\alpha} M \to \operatorname{cofib}(\alpha)$$

which determines a long exact sequence of abelian groups

$$0 \to \pi_{2k} M \to \pi_{2k} \operatorname{cofib}(\alpha) \to \mathbf{Z} \xrightarrow{\eta} \pi_{2k-1} M \to \pi_{2k-1} \operatorname{cofib}(\alpha) \to 0$$

It follows that $\pi_{2k-1} \operatorname{cofib}(\alpha)$ is the direct sum $F \oplus T/\eta$, and that $\pi_{2k} \operatorname{cofib}(\alpha)$ is an extension of a finite index subgroup of $m\mathbf{Z} \subseteq \mathbf{Z}$ (where m is the order of η) by $\pi_{2m} M$. Such an extension is automatically split, so that $\pi_{2k} \operatorname{cofib}(\alpha) \simeq \operatorname{Hom}(F, \mathbf{Z}) \oplus m\mathbf{Z}$.

Let M_α denote the result of surgery along α, so that we have a fiber sequence

$$M_\alpha \to \mathrm{cofib}(\alpha) \to \Sigma^{2k}\mathbf{Z}$$

and therefore a long exact sequence of homotopy groups

$$0 \to \pi_{2m} M_\alpha \to \pi_{2m} \mathrm{cofib}(\alpha) \xrightarrow{\phi} \mathbf{Z} \to \pi_{2m-1} M_\alpha \to \pi_{2m-1} \mathrm{cofib}(\alpha) \to 0$$

Note that the restriction of ϕ to $\pi_{2m} M = \mathrm{Hom}(F, \mathbf{Z})$ is dual to the map $\mathbf{Z} \to F$ determined by η. Since η was chosen to be a torsion element, this map is zero. Consequently, ϕ factors as a composition

$$\pi_{2k} \mathrm{cofib}(\alpha) \to m\mathbf{Z} \xrightarrow{\phi_0} \mathbf{Z}.$$

We will need the following:

($*$) The map ϕ_0 vanishes.

Assuming ($*$) for the moment, we deduce that $\pi_{2k-1} M_\alpha$ is an extension of $\pi_{2k-1} \mathrm{cofib}(\alpha)$ by the group $\mathbf{Z}$. In particular, the torsion subgroup of $\pi_{2k-1} M_\alpha$ injects into the torsion subgroup T/η of $\mathrm{cofib}(\alpha)$. We conclude that $\pi_{2k-1} M_\alpha$ has a smaller torsion subgroup than $\pi_{2k-1} M$ (though it has a larger free part).

Repeating this procedure, we can reduce to the case where $T = 0$. Then $\pi_{2k-1} M$ is a free abelian group F, and $\pi_{2k} M \simeq \mathrm{Hom}(F, \mathbf{Z})$ is the dual abelian group. Since F is free, we can choose a map $\Sigma^k F \to M$ which induces the identity map

$$F \simeq \pi_{2k-1} \Sigma^{2k-1} F \to \pi_{2k-1} M \simeq F.$$

Note that $\Sigma^{4k-1} Q^q(\Sigma^{2k-1} F)$ is a connected spectrum, so that $q|\Sigma^k F$ is automatically nullhomotopic. Any choice of nullhomotopy exhibits F as a Lagrangian in (M, q), so that (M, q) is nullcobordant.

It remains to prove ($*$). For this, we can tensor everything with $\mathbf{Q}$. In this case, the map $\alpha_{\mathbf{Q}} : \Sigma^{2k-1} \mathbf{Q} \to M_{\mathbf{Q}}$ is the zero map, so that the surgery data is given by a choice of nullhomotopy of $q|(\Sigma^{2k-1}) = 0$: that is, by an element $p \in \pi_0 \Sigma^{4k-2} Q^q(\Sigma^{2k-1} \mathbf{Q})$. We can identify ϕ_0 with the underlying bilinear form on $\mathbf{Q}$. Since $2k-1$ is odd, this bilinear form is skew-symmetric, and therefore vanishes. □

Remark 2. The above argument works if we replace $\mathbf{Z}$ by any Dedekind ring: for example, the ring of integers in a number field.

The next case is more difficult.

Proposition 3. *The quadratic groups $L^q_n(\mathbf{Z})$ vanish when n is congruent to 3 modulo 4.*

Proof. Without loss of generality, $n = -1$: that is, we are studying the L-groups of the pair $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}, \Sigma Q^q)$. Let (M, q) be a Poincare object. Arguing as in the proof of Proposition 1, we can use surgery below the middle dimension to assume that

$$\pi_* M \simeq \begin{cases} F \oplus T & \text{if } * = 0 \\ \mathrm{Hom}(F, \mathbf{Z}) & \text{if } * = 1 \\ 0 & \text{otherwise.} \end{cases}$$

where F is a free abelian group and T is a finite abelian group. These isomorphisms determine a map $\alpha : F \to M$, which is well-defined up to homotopy. Performing surgery along α, we can reduce to the case where $F = 0$. Then $M \simeq T$ can be identified with a finite abelian group, concentrated in degree zero.

Let A be any finite abelian group. Let us spell out what it means to endow A with the structure of a quadratic object of $(\mathrm{LMod}^{\mathrm{fp}}_{\mathbf{Z}}, \Sigma Q^q)$. Choose a free abelian group Λ and a surjection $\Lambda \to A$, with kernel Λ_0. We then have a fiber sequence

$$Q^q(A) \to Q^q(\Lambda) \to Q^q(\Lambda_0) \times_{B(\Lambda_0, \Lambda_0)} B(\Lambda_0, \Lambda).$$

The spectra $B(\Lambda_0, \Lambda)$ and $B(\Lambda_0, \Lambda_0)$. Since $\pi_{-1} Q^q(\Lambda) \simeq 0$, we deduce that $\pi_{-1} Q^q(A)$ can be identified with the cokernel of the map of abelian groups

$$\pi_0 Q^q(\Lambda) \to \pi_0(Q^q(\Lambda_0) \times_{B(\Lambda_0,\Lambda_0)} B(\Lambda, \Lambda_0)).$$

We have a fiber sequence of spectra

$$Q^q(\Lambda_0) \times_{B(\Lambda_0,\Lambda_0)} B(\Lambda, \Lambda_0) \to Q^q(\Lambda_0) \times B(\Lambda, \Lambda_0) \to B(\Lambda_0, \Lambda_0)$$

Since $\pi_1 B(\Lambda_0, \Lambda_0) \simeq 0$, we can identify $\pi_0(Q^q(\Lambda_0) \times_{B(\Lambda_0,\Lambda_0)} B(\Lambda, \Lambda_0))$ with the fiber product of abelian groups

$$\pi_0 Q^q(\Lambda_0) \times_{\pi_0 B(\Lambda_0,\Lambda_0)} \pi_0 B(\Lambda, \Lambda_0).$$

Recall that $\pi_0 Q^q(\Lambda_0)$ can be identified with the abelian group of quadratic forms $f : \Lambda_0 \to \mathbf{Z}$. Every such quadratic form determines symmetric bilinear form $b : \Lambda_0 \times \Lambda_0 \to \mathbf{Z}$, hence a map $b_{\mathbf{Q}} : \Lambda \times \Lambda \to \mathbf{Q}$. We can identify $\pi_0 Q^q(\Lambda_0) \times_{\pi_0 B(\Lambda_0,\Lambda_0)} \pi_0 B(\Lambda, \Lambda_0)$ with the subgroup of $\pi_0 Q^q(\Lambda_0)$ consisting of those quadratic forms f such that b is integral on $\Lambda \times \Lambda_0$. We have proven the following:

$(*)$ If we are given an exact sequence of abelian groups

$$0 \to \Lambda_0 \to \Lambda \to A \to 0$$

as above, then $\pi_{-1} Q^q(A)$ can be identified with the quotient X/X_0, where X is the abelian group of quadratic forms on Λ_0 whose associated bilinear form is integral on $\Lambda_0 \times \Lambda$, and X_0 is the abelian group of quadratic forms on Λ.

Note that every element of X/X_0 determines a well-defined quadratic form $\epsilon : A \to \mathbf{Q}/\mathbf{Z}$. Conversely, suppose that $\epsilon : A \to \mathbf{Q}/\mathbf{Z}$ is any quadratic form. Write $A = \bigoplus (\mathbf{Z}/n_i\mathbf{Z}) x_i$, and take $\Lambda = \bigoplus \mathbf{Z}\overline{x}_i$ and $\Lambda_0 = \bigoplus (n_i \mathbf{Z})\overline{x}_i$. Choose rational numbers $\delta_i \in \mathbf{Q}$ lifting $\epsilon(x_i) \in \mathbf{Q}/\mathbf{Z}$, and rational numbers $\beta_{i,j} \in \mathbf{Q}$ lifting $\epsilon(x_i + x_j) - \epsilon(x_i) - \epsilon(x_j) \in \mathbf{Q}/\mathbf{Z}$ for $i < j$. Define a quadratic form $q : \Lambda \to \mathbf{Q}$ so that $q(\overline{x}_i) = \delta_i$ and $q(\overline{x}_i + \overline{x}_j) = q(\overline{x}_i) + q(\overline{x}_j) + \beta_{i,j}$. Let b be the associated bilinear form. By construction, the composite map

$$\Lambda \times \Lambda \xrightarrow{b} \mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$$

factors through $A \times A$, and therefore vanishes on $\Lambda_0 \times \Lambda$. Thus b is integral on $\Lambda_0 \times \Lambda$, and therefore on $\Lambda_0 \times \Lambda_0$. We claim that q is integral on Λ_0. Since b is integral on $\Lambda_0 \times \Lambda_0$, it suffices to check on generators: that is, we need $q(n_i \overline{x}_i) \in \mathbf{Z}$. Note that $q(n_i x_i) = n_i^2 \delta_i$ has image $n_i^2 \epsilon(x_i) = \epsilon(n_i x_i) = 0$ in $\mathbf{Q}/\mathbf{Z}$. It follows that every quadratic form $\epsilon : A \to \mathbf{Q}/\mathbf{Z}$ can be lifted to an element of X, which is obviously unique up to an element of X_0. We therefore have the following more invariant description of $\pi_{-1} Q^q(A)$:

$(*')$ Let A be a finite abelian group. Then $\pi_{-1} Q^q(A)$ can be identified with the abelian group of quadratic forms $\epsilon : A \to \mathbf{Q}/\mathbf{Z}$.

Now suppose we are given a Poincare object (A, ϵ). We will prove that (A, ϵ) is nullbordant using induction on the order of A. Let $a \in A$ be an element having order $n > 1$, and identify a with a map $\alpha : \mathbf{Z} \to A$. To perform surgery along α, we need to choose a nullhomotopy of the restriction of ϵ to $\mathbf{Z}$. Unwinding the definitions, this amounts to choosing a quadratic form $q : \mathbf{Z} \to \mathbf{Q}$ such that $q(1) = \epsilon(a)$. In this case, the cofiber of α has homotopy groups

$$\pi_* \operatorname{cofib}(\alpha) = \begin{cases} A/a & \text{if } i = 0 \\ n\mathbf{Z} & \text{if } i = 1. \end{cases}$$

The result of the surgery is a new $\mathbf{Z}$-module spectrum A_α which fits into a fiber sequence

$$A_\alpha \to \operatorname{cofib}(\alpha) \to \mathbf{Z}.$$

In particular, we have an exact sequence

$$n\mathbf{Z} \stackrel{\psi}{\to} \mathbf{Z} \to \pi_0 A_\alpha \to A/a \to 0.$$

Unwinding the definitions, we see that ψ is determined by the bilinear form determines by q: in other words, we have $\psi(n) = 2nq(1)$. Note that we are free to adjust our nullhomotopy by adding an integral bilinear form to q: in other words, we are free to adjust $\psi(n)$ by multiples of $2n$. We may therefore arrange that the absolute value of $\psi(n)$ is $\leq n$.

If $\psi(n) = 0$, then the exact sequence above shows that $\pi_0 A_\alpha$ is an extension of A/a by the group of integers $\mathbf{Z}$. It follows that the torsion subgroup of $\pi_0 A_\alpha$ injects into A/a. As in the first step, we can perform surgery on A_α to kill the torsion free-part without changing the torsion part: it follows that (A, ϵ) is cobordant to a pair (A', ϵ') where A' is isomorphic to a subgroup of A/a, and is therefore smaller than A. If $\psi(n) \neq 0$, we have an exact sequence

$$0 \to \mathbf{Z}/\psi(n)\mathbf{Z} \to \pi_0 A_\alpha \to A/a \to 0.$$

It follows that A_α can be identified with a finite abelian group of order $\frac{|\psi(n)|}{n}|A|$. This is smaller than the order of A unless $|\psi(n)| = n$. Since $\psi(n) = 2nq(1)$, we get $q(1) \equiv \frac{1}{2}$ (modulo $\mathbf{Z}$), so that $\epsilon(a) = \frac{1}{2}$.

In the above discussion, we choose a to be an arbitrary nonzero element of A. It follows that we can simplify A by means of surgery unless the quadratic form $\epsilon : A \to \mathbf{Q}/\mathbf{Z}$ takes the value $\frac{1}{2}$ on every nonzero element of A. In this case, $2\epsilon = 0$. Since (A, ϵ) is Poincare, ϵ is nondegenerate on A so that A is annihilated by multiplication by 2. We can therefore identify A with a finite-dimensional vector space over $\mathbf{F}_2$ and ϵ with an anisotropic quadratic form on A with values in $\frac{1}{2}\mathbf{Z}/\mathbf{Z} \simeq \mathbf{F}_2$. We have seen that if $A \neq 0$, then A must be isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, with ϵ given by

$$\epsilon(a) = \begin{cases} 0 \text{ if } a = 0 \\ \frac{1}{2} & \text{if } a \neq 0. \end{cases}$$

We now complete the proof by explicitly exhibiting a nullhomotopy of (A, ϵ). To do this, we will write (A, ϵ) as the "boundary" of a quadratic object: that is, we will choose a lattice Λ_0 equipped with an integral quadratic form q_0 which induces an injection $\Lambda_0 \hookrightarrow \Lambda_0^\vee$, so that $A \simeq \Lambda_0^\vee/\Lambda_0$ and ϵ is the reduction modulo $\mathbf{Z}$ of the induced quadratic form $\Lambda_0^\vee \to \mathbf{Q}$. For this, we take Λ_0 to be the D_4-lattice. That is, Λ_0 is the free abelian group on generators w, x, y, and z, satisfying

$$q(w) = q(x) = q(y) = q(z) = 1$$

$$b(w, x) = b(w, y) = b(w, z) = -1$$

$$b(x, y) = b(x, z) = b(y, z) = 0.$$

The cosets of Λ_0 in its dual (which we identify with a subset of $\Lambda_0 \otimes \mathbf{Q}$) are represented by 0, $\frac{x+y}{2}$, $\frac{x+z}{2}$, and $\frac{y+z}{2}$. Note that

$$q(\frac{x+y}{2}) = q(\frac{x}{2}) + q(\frac{y}{2}) + b(\frac{x}{2}, \frac{y}{2}) = \frac{1}{4} + \frac{1}{4} + 0 = \frac{1}{2},$$

and similarly for the other nontrivial cosets. □