L-Groups of Fields (Lecture 13)

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Our goal in this section is to carry out some calculuations of L-groups in simple cases. We begin with the following observation:

Proposition 1. *Let R be an associative ring with involution. Then the L-groups of R (symmetric or quadratic) are 4-periodic. That is, there are canonical isomorphisms*

$$L^s_n(R) \simeq L^s_{n+4}(R) \qquad L^q_n(R) \simeq L^q_{n+4}(R)$$

Proof. Let $\mathcal{C} = \mathrm{LMod}^{\mathrm{fp}}_R$. Since $\mathcal{C}$ is a stable ∞-category, the suspension functor is an equivalence from $\mathcal{C}$ to itself. Let B be the symmetric bilinear functor given by $B(M,N) = \mathrm{Mor}_{R-R}(M \wedge N, R)$. Then $B(\Sigma M, \Sigma N) \simeq \Sigma^{-2} B(M,N)$. Here $\Sigma^{-2} B$ is also a symmetric bilinear functor, where the symmetric group Σ_2 acts on B and also on the desuspenion functor Σ^{-2} by permuting the suspension coordinates. Because the "swap" map on the sphere $S^2 = S^1 \wedge S^1$ reverses orientation, this second action is nontrivial: it acts by a sign. However, the square of this action is trivial. Consequently, we have a equivalence $B(\Sigma^2 M, \Sigma^2 N) \simeq \Sigma^{-4} B(M,N)$, compatible with the action of Σ_2 (where Σ_2 does not act on the desuspension functor Σ^{-4}). Consequently, the double suspension map $\mathcal{C} \to \mathcal{C}$ determines equivalences

$$(\mathcal{C}, Q^s) \simeq (\mathcal{C}, \Sigma^{-4} Q^s) \qquad (\mathcal{C}, Q^q) \simeq (\mathcal{C}, \Sigma^{-4} Q^q).$$

□

Remark 2. Suppose that $2 = 0$ in R. Then we can ignore signs. The proof of Proposition 1 then shows that the L-groups of R are 2-periodic.

Let us now restrict our attention to the case where R is a (commutative) field k, equipped with the trivial involution. Note that if the characteristic of k is different from 2, then there is no difference between symmetric and quadratic L-theory. We will confine our attention to quadratic L-theory in what follows.

Proposition 3. *Let k be a field. Then the odd-dimensional quadratic L-groups $L^q_{-2m-1}(k)$ are trivial.*

Proof. Let (V, q) be a Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_k, \Sigma^{2m+1} Q^q)$. We wish to show that (V, q) is nullcobordant. In the last lecture, we saw that we can reduce to the case where V is k-connective. The nondegeneracy of q gives an isomorphism $V \simeq \Sigma^{2m+1} \mathbb{D}(V)$. Since we are working over a field, this has concrete consequences: for every integer i, $\pi_i V$ is the k-linear dual of $\pi_{2m+1-i}(V)$. In particular, the homotopy groups $\pi_i V$ vanish for $i \notin \{m, m+1\}$. Let $W = \pi_m V$ so that $W^\vee \simeq \pi_{m+1} V$. Let $W[m] \in \mathrm{LMod}^{\mathrm{fp}}_k$ denote the module given by W, placed in degree k. Since k is a field, W is free as a k-module. We may therefore construct a map

$$\alpha : W[m] \to V$$

which induces the identity map

$$W \simeq \pi_m W[m] \to \pi_m V \simeq W.$$

Note that $\Sigma^{2m+1} Q^q(W[k]) \simeq (W \otimes_k W)[1]_{h\Sigma_2}$ is connected, so $q | W[m]$ is automatically nullhomotopic. Any choice of nullhomotopy exhibits $W[m]$ as a Lagrangian in V. □

Proposition 4. *Let k be a field of characteristic different from 2. Then the L-groups $L^q_{-4m-2}(k)$ are trivial. (If k has characteristic 2, then $L^q_{-4m-2}(k) \simeq L^q_0(k)$ by Remark 2.)*

Proof. Let (M, q) be a Poincare object of $(\mathrm{LMod}^{\mathrm{fp}}_k, \Sigma^{4m+2} Q^q)$. The results of the last lecture show that we can assume that $M = V[2m+1]$ for some vector space V over k. Let $B(V, V)$ denote the k-vector space of symmetric bilinear forms on V (regarded as a spectrum concentrated in a single degree). Then $\Sigma^{4m+2} Q^q(M) = \Sigma^{4m+2}(\Sigma^{-4m-2} B(V, V))_{h\Sigma_2}$. Here we can ignore the distinction between invariants and coinvariants (since 2 is invertible in k). However, we cannot ignore the fact that Σ_2 acts nontrivially on the suspension coordinates. The upshot is that $\Sigma^{4m+2} Q^q(M)$ is the Eilenberg-MacLane spectrum corresponding to the vector space of *skew-symmetric* bilinear forms $b : V \times V \to k$. Since (M, q) is a Poincare object, the corresponding skew-symmetric form is nondegenerate. It follows from elementary linear algebra that the dimension of V must be even, and that V admits a subspace $L \subseteq V$ of such that $b|(L \times L)$ is trivial $\dim(V) = 2\dim(L)$. Then L is a Lagrangian in V, so that (M, q) is nullcobordant.

Here is a slight variant on the above argument: if $V \neq 0$, then by skew-symmetry the bilinear form b vanishes on the one-dimensional subspace generated by any nonzero element $v \in V$. We can therefore perform surgery to reduce the dimension of V. Repeat until $V \simeq 0$.) □

In view of Propositions 1, 3, and 4, the calculation of the (quadratic) L-groups of fields reduces to the problem of understanding the group $L^q_0(k)$. This is an interesting classical invariant.

Definition 5. Let k be a field. A *quadratic space* over k is a pair (V, q), where V is a finite-dimensional vector space over k and $q : V \to k$ is a quadratic form. That is, q satisfies

$$q(ax) = a^2 q(x) \qquad q(x+y) = q(x) + q(y) + b(x, y)$$

for some bilinear form $b : V \times V \to k$. We say that q is *nondegenerate* if b is nondegenerate.

Example 6. Let k be any field. There is a quadratic space $H = (k^2, q)$ over k, where q is given by the formula $q(a, b) = ab$. We refer to H as the *hyperbolic plane*.

There is an evident direct sum operation on quadratic spaces: given a pair of quadratic spaces (V, q) and (V', q'), we define $(V, q) \oplus (V', q')$ to be $(V \oplus V', q \oplus q')$, where $q \oplus q' : V \oplus V' \to k$ is given by the formula

$$(q \oplus q')(v, v') = q(v) + q'(v').$$

Remark 7. Let (V, q) be a nondegenerate quadratic space over a field k. Suppose we are given a nonzero element $x \in V$ such that $q(x) = 0$. Since the associated bilinear form b is nondegenerate, we can choose $y \in V$ with $b(x, y) = 1$. Note that $b(x, x) = q(2x) - q(x) - q(x) = 2q(x) = 0$. It follows that $q(y + ax) = q(y) + ab(y, x) + q(ax) = q(y) + a$. In particular, $q(y - q(y)x) = 0$. Replacing y by $y - q(y)x$, we can reduce to the case where $q(y) = 0$. Then if V_0 denotes the subspace of V generated by x and y, then we have an isomorphism $(V_0, q|V_0) \simeq H$. In particular, q is nondegenerate on V_0 and we therefore have a decomposition $(V, q) \simeq H \oplus (V_1, q|V_1)$, where V_1 is the orthogonal complement of V_0.

More generally, if we are given a subspace $W \subseteq V$ of dimension a such that $q|W = 0$, we can apply this argument repeatedly to obtain a decomposition $(V, q) \simeq H^{\oplus a} \oplus (V', q')$.

Definition 8. Let k be a field. We say that two nondegenerate quadratic spaces (V, q) and (V', q') are *stably equivalent* if $(V, q) \oplus H^{\oplus a}$ is isomorphic to $(V', q') \oplus H^{\oplus b}$ for some integers a and b. The collection of stable equivalences classes of nondegenerate quadratic spaces over k is called the *Witt group* of k. We will denote it by $W(k)$ (not to be confused with the *ring of Witt vectors over* k).

The set $W(k)$ evidently has the structure of a commutative monoid under direct sum. In fact, this monoid structure is a group: for any nondegenerate quadratic space (V, q) where V has dimension d, the sum $(V, q) \oplus (V, -q)$ has an isotopic subspace of dimension d (the image of V under the diagonal map $V \to V \oplus V$) and is therefore isomorphic to $H^{\oplus d}$ by Remark 7.

Remark 9. Let (V, q) be any nondegenerate quadratic space over k. Using Remark 7 repeatedly, we deduce that (V, q) is isomorphic to a direct sum $(V', q') \oplus H^{\oplus d}$ for some integer d, where (V', q') is *anisotropic*: that is, q' does not vanish on any nonzero element of V'. Consequently, every class in the Witt group $W(k)$ can be represented by an anisotropic quadratic space (V, q). In fact, this representative is *unique* up to isomorphism. This is a consequence of the Witt cancellation theorem, which asserts that if we have an isomorphism of nondegenerate quadratic spaces

$$(V, q) \oplus (V'', q'') \simeq (V', q') \oplus (V'', q''),$$

then (V, q) and (V', q') must already be isomorphic.

Let (V, q) be a nondegenerate quadratic space over a field k. Viewing V as a chain complex over k concentrated in degree zero, we can think of (V, q) as a Poincare object of $(\mathrm{LMod}_k^{\mathrm{fp}}, Q^q)$. This construction determines a map $W(k) \to L_0^q(k)$.

Proposition 10. *Let k be a field. Then the map $\phi : W(k) \to L_0^q(k)$ is an isomorphism of abelian groups.*

Proof. We have already seen that ϕ is surjective (using surgery below the middle dimension). Let us show that ϕ is injective. Let (V, q) be a quadratic space over k, and suppose that there exists a Lagrangian in V (as a Poincare object of $(\mathrm{LMod}_k^{\mathrm{fp}}, Q^q)$. Denoting this Lagrangian by L, we have a fiber sequence of spectra

$$L \xrightarrow{\alpha} V \to \mathrm{cofib}(\alpha)$$

which is self-dual (with the duality on V determined by q). In particular, we have a self-dual short exact sequence of vector spaces

$$0 \to (\mathrm{Im}\, \pi_0 L \to V) \to V \to (\mathrm{Im}\, V \to \pi_0 \, \mathrm{cofib}(\alpha)) \to 0.$$

The self-duality implies that the dimensions of the outer two vector spaces are the same, so that the dimension of V is twice as large as the dimension of $W = \mathrm{Im}(\pi_0 L \to V)$. The map $W \to V$ factors through L, so $q|W = 0$. Using Remark 7, we deduce that V is isomorphic to a direct sum of hyperbolic planes so that (V, q) is equivalent to zero in the Witt group $W(k)$. □

Example 11. Let $k = \mathbf{F}_2$ be the finite field with two elements. Let (V, q) be a nondegenerate quadratic space over k. Then the dimension of V must be even (since the symmetric bilinear form b is also a nondegenerate skew-symmetric bilinear form). Suppose that V is anisotropic: then $q(v) = 1$ for every nonzero element $v \in V$. It follows that if $v, w \in W$ are distinct and nonzero, then $b(v, w) = q(v + w) - q(v) - q(w) = 1$. If $u, v, w \in W$ are linearly independent, we get

$$1 = b(u, v + w) = b(u, v) + b(u, w) = 0.$$

Thus any nontrivial anisotropic quadratic space must be of dimension 2. There is such a space (V, q): take $V = \mathbf{F}_2 \oplus \mathbf{F}_2$, and q to be given by the formula

$$q(a, b) = a^2 + ab + b^2.$$

It follows from the Witt cancellation theorem that (V, q) determines a nontrivial element of $W(k)$ (this can also be deduced by evaluating some of the invariants introduced below). We therefore have an isomorphism $W(k) \simeq \mathbf{Z}/2\mathbf{Z}$.

To any nondegenerate quadratic space (V, q) over $k = \mathbf{F}_2$, we can associate an invariant in the group $W(k = \mathbf{Z}/2\mathbf{Z}$. This is called the *Arf invariant* of (V, q). It can be described concretely as follows: the Arf invariant of q is 0 if q takes the value 0 more often than 1 (that is, if the set $q^{-1}\{0\} \subseteq V$ is larger than the set $q^{-1}\{1\} \subseteq V$), and takes the value 1 otherwise. A more conceptual description of this invariant is given below.

Example 12. Let k be an algebraically closed field. Any two nondegenerate quadratic spaces (V, q) over k of the same dimension are isomorphic. It follows that $W(k)$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ if the characteristic of k is different from 2, and is trivial if the characteristic of k is equal to 2 (since any nondegenerate quadratic space must be even dimension in the latter case).

Example 13. Let k be the field of real numbers (or any real-closed field). Then Sylvester's invariance of signature theorem gives an isomorphism of abelian groups $W(k) \simeq \mathbf{Z}$, which carries a nondegenerate quadratic space (V, q) to the signature $\sigma(q) \in \mathbf{Z}$.

Remark 14. Let k be a field of characteristic $\neq 2$. For every nonzero element $a \in k$, we have a nondegenerate quadratic form $q : k \to k$ given by $x \mapsto ax^2$. We denote the image of this element in $W(k)$ by $\langle a \rangle$. These elements generate $W(k)$ under addition, because any nondegenerate quadratic space (V, q) has an orthogonal basis. It is possible to explicitly write down a set of relations between these generators, and thereby obtain a presentation for $W(k)$.

We now describe some invariants that can help get a handle on the structure of a Witt ring $W(k)$. We first note that since the hyperbolic plane H has dimension 2, every element of $W(k)$ has a well-defined dimension modulo 2. This yields a group homomorphism

$$d : W(k) \to \mathbf{Z}/2\mathbf{Z}.$$

The map d is surjective when k has characteristic different from 2, and is the zero map when k has characteristic 2. Let I denote the kernel of d.

Suppose we are given an element of $I \subseteq W(k)$. We can represent this element by a nondegenerate quadratic space (V, q) of even dimension over k. We define the *Clifford algebra* $\mathrm{Cl}(V, q)$ to be the quotient of the free associative k-algebra on V by the relations

$$x^2 = q(x)$$

for $x \in V$. This Clifford algebra has a canonical $\mathbf{Z}/2\mathbf{Z}$-grading

$$\mathrm{Cl}(V, q) \simeq \mathrm{Cl}_0(V, q) \oplus \mathrm{Cl}_1(V, q),$$

where we take the elements of V to have degree 1. If $V \neq 0$, then one can show that the center of $\mathrm{Cl}_0(V, q)$ is a rank 2 étale extension of k: that is, it is either isomorphic to $k \times k$ or to a separable quadratic extension field k' of k. This extension of k determines a map $\mathrm{Gal}(\overline{k}/k) \to \mathbf{Z}/2\mathbf{Z}$, (which is the zero map if and only if the center is isomorphic to $k \times k$). The formation of this invariant determines a group homomorphism

$$\psi : I \to \mathrm{H}^1(\mathrm{Gal}(\overline{k}/k); \mathbf{Z}/2\mathbf{Z}).$$

The image of a quadratic space (V, q) under this map is called the *discriminant* of (V, q). When the characteristic of k is different from 2, we can identify the discriminant with an element in $k^\times/(k^\times)^2$ (using Kummer theory). When k has characteristic 2, we can identify the discriminant with an element of the cokernel of the map

$$k \xrightarrow{x \mapsto x - x^2} k$$

(using Artin-Schreier theory). When $k = \mathbf{F}_2$, we recover the Arf invariant discussed above.

Let $J \subseteq W(k)$ denote the kernel of the map ψ defined above. Elements of J can be represented by quadratic spaces (V, q) of even dimension such that the center of $\mathrm{Cl}_0(V, q)$ splits as a product $k \times k$. It follows that $\mathrm{Cl}_0(V, q)$ itself splits as a product of two factors. One can show that each of these factors is a central simple algebra over k, and determines a 2-torsion element in the Brauer group of k. Let us assume that k has characteristic different from 2, so we can identify $\mathbf{Z}/2\mathbf{Z}$ with the subset $\{1, -1\} \subseteq k^\times$. Extracting these Brauer invariants gives a homomorphism

$$J \to \mathrm{H}^2(\mathrm{Gal}(\overline{k}/k); \mathbf{Z}/2\mathbf{Z}).$$

This pattern continues. If the characteristic of k is different from 2, then the Witt group $W(k)$ actually has the structure of a *ring* (given symmetric bilinear forms on vector spaces V and W, we obtain a symmetric bilinear form on $V \otimes W$). The map $d : W(k) \to \mathbf{Z}/2\mathbf{Z}$ is a ring homomorphism, so that $I \subseteq W(k)$ is an ideal. The following is a deep result of Voevodsky:

Theorem 15 (The Milnor Conjecture)**.** *If k is a field of characteristic different from 2, there are canonical isomorphisms*

$$I^m/I^{m+1} \simeq \mathrm{H}^m(\mathrm{Gal}(\overline{k}/k); \mathbf{Z}/2\mathbf{Z})$$