Surgery Below the Middle Dimension (Lecture 12)

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In the previous lecture, we discussed the process of *surgery*. If $\mathcal{C}$ is a stable ∞-category equipped with a nondegenerate functor Q and we are given a quadratic object (X, q), a map $\alpha : X' \to X$, and a nullhomotopy of $q|X'$, then we can construct a new quadratic (X_α, q_α) by "surgery along α". Our goal in this lecture is to show how this construction can be used to simplify a quadratic object.

First, we need to review a bit of terminology. Let X be a spectrum and $n \in \mathbf{Z}$ an integer. We say that X is *n-connective* if the homotopy groups $\pi_i X$ vanish for $i < n$. We say that X is *connective* if X is 0-connective: that is, if the negative homotopy groups of X are trivial. The collection of connective spectra is stable under smash products and homotopy colimits.

Let R be an A_∞-ring. Suppose M and N are right and left R-module spectra, respectively. Then $M \wedge_R N$ is the given by the geometric realization of a simplicial spectrum, whose nth term is an iterated smash product

$$M \wedge R \wedge \cdots \wedge R \wedge N.$$

If M, R, and N are connective, then $M \wedge_R N$ is also connective.

For every perfect left module M over R, we let $\mathbb{D}(M)$ denote the mapping spectrum $\mathrm{Mor}_R(M, R)$. This is a perfect right module over R (which we can identify with a left R-module in the special case where R has an involution). We will say that M has *projective amplitude* $\leq n$ if $\mathbb{D}(M)$ is $(-n)$-connective. Let N be any other left R-module. We then have a canonical equivalence

$$\mathrm{Mor}_R(M, N) \simeq \mathbb{D}(M) \wedge_R N.$$

If M has projective amplitude $\leq n$ and both R and N are connective, then the above discussion shows that $\mathbb{D}(M) \wedge_R N$ is $(-n)$-connective. In particular, if M has projective amplitude ≤ 0, then $\mathbb{D}(M) \wedge_R N$ is connective.

Proposition 1. *Let R be a connective A_∞-ring. Let M be a perfect left R-module which is connective and of projective amplitude ≤ 0. Then M is a direct summand of R^n for some n. In this case, we will say that M is projective.*

Proof. Using the fact that M is perfect and that the negative homotopy groups $\pi_i M$ vanish, one can show that $\pi_0 M$ is a finitely generated R-module. We can therefore choose a map $R^n \to M$ which is surjective on π_0. We have a fiber sequence

$$M' \to R^n \to M,$$

where M' is also projective. This gives another fiber sequence of spectra

$$\mathrm{Mor}_R(M, M') \to \mathrm{Mor}_R(M, R^n) \to \mathrm{Mor}_R(M, M).$$

Since M' is connective and M is of projective amplitude ≤ 0, the mapping spectrum $\mathrm{Mor}_R(M, M')$ is connective. Using the long exact sequence of homotopy groups, we see that the map $\pi_0 \mathrm{Mor}_R(M, R^n) \to \pi_0 \mathrm{Mor}_R(M, M)$ is surjective. This means that there is a map $\phi : M \to R^n$ lifting the identity map $\mathrm{id}_M : M \to M$, so that M is a direct summand of R^n. $\square$

Variant 2. In the situation of Proposition 1, if M is k-connective and of projective amplitude $\leq k$, then the same argument shows that M is a summand of $\Sigma^k R^n$ for some n.

Let us now suppose that R is a connective A_∞-ring equipped with an involution σ, and let Q^{quad} be the quadratic functor on $\text{LMod}_R^{\text{fp}}$ given by the formula $Q^{\text{quad}}(M) = \text{Mor}_{R-R}(M \wedge M, R)_{h\Sigma_2}$. We would like to study the L-groups $L_n^{\text{quad}}(R)$. We begin by studying the case where n is even, so we can write $n = -2k$. Then $L_n^{\text{quad}}(R) = L_0(\text{LMod}_R^{\text{fp}}, \Sigma^{2k} Q^{\text{quad}})$.

Suppose we are given a quadratic object (M, q) (so that q lies in the zeroth space of the spectrum $\Sigma^{2k}(\mathbb{D}(M) \wedge_R \mathbb{D}(M))_{h\Sigma_2}$). Note that if $M = \Sigma^m R$, then $\Sigma^{2k} Q^{\text{quad}}(M)$ can be identified with the homotopy coinvariants of Σ_2 acting on the spectrum $\Sigma^{2k-2m}(R)$ (through a mixture of the involution σ on R and a permutation of suspension coordinates). If $m < k$, then the homotopy groups of this spectrumvanish for $i \leq 0$. This remains true after passing to homotopy coinvariants: that is, the spectrum $\Sigma^{2k} Q^{\text{quad}}(M)$ is connected (even simply connected), so there exists a nullhomotopy of q.

Now let (M, q) be an arbitrary quadratic object of $(\text{LMod}_R^{\text{fp}}, \Sigma^{2k} Q^{\text{quad}})$. Suppose we are given a homotopy class $\eta \in \pi_m M$. Then η determines a map of R-module spectra $\alpha : \Sigma^m R \to M$. If $m < k$, then the above calculation shows that $q|\Sigma^m R$ is automatically nullhomotopic. Choosing a nullhomotopy, we can perform surgery to obtain a new quadratic object (M_α, q_α). Let us study the homotopy groups of resulting object.

Consider the map $\beta : M \to \Sigma^{2k}\mathbb{D}(\Sigma^m R) = \Sigma^{2k-m} R$ determined by our choice of nullhomotopy of $q|\Sigma^m R$. We have an exact sequence

$$\pi_{i+1}\Sigma^{2k-m} R \to \pi_i \operatorname{fib}(\beta) \to \pi_i M \to \pi_i \Sigma^{2k-m} R.$$

Since R is connective, the homotopy group $\pi_i \Sigma^{2k-m} R = \pi_{i+m-2k} R$ vanishes if $i + m < 2k$. Similarly, $\pi_{i+1}\Sigma^{2k-m} R \simeq 0$ if $i + m + 1 < 2k$. If $m < k$, then we conclude that $\pi_i \operatorname{fib}(\beta) \simeq \pi_i M$ for $i \leq k$.

We have another exact sequence

$$\pi_i \Sigma^m R \to \pi_i \operatorname{fib}(\beta) \to \pi_i M_\alpha \to \pi_{i-1} \Sigma^m R.$$

Since R is connective, we conclude that $\pi_i M_\alpha \simeq \pi_i \operatorname{fib}(\beta) \simeq \pi_i M$ for $i < m$, and that $\pi_m M_\alpha$ is the quotient of $\pi_m \operatorname{fib}(\beta) \simeq \pi_m M$ by the submodule generated by η. We can summarize our discussion as follows:

Lemma 3. *Let (M, q) be a quadratic object of $(\text{LMod}_R^{\text{fp}}, \Sigma^{2k} Q^{\text{quad}})$. Suppose that we are given a class $\eta \in \pi_m M$, where $m < k$. Then (M, q) is cobordant to another Poincare object (M', q'), where $\pi_i M' \simeq \pi_i M$ for $i < m$, and $\pi_m M'$ is the quotient of $\pi_m M$ by the submodule generated by η.*

Let (M, q) be as in Lemma 3, and assume that M is nonzero. Since R is connective, there exists a smallest integer m such that $\pi_m M$ is nonzero. Moreover, the finiteness of M guarantees that $\pi_m M$ is a finitely generated module over $\pi_0 R$. If $m < k$, we can apply Lemma 3 repeatedly to obtain another quadratic object (M', q'), with $\pi_i M' \simeq 0$ for $i \leq m$. Applying *this* argument repeatedly, we obtain the following result:

Proposition 4. *Let (M, q) be a Poincare object of $(\text{LMod}_R^{\text{fp}}, \Sigma^{2k} Q^{\text{quad}})$. Then (M, q) is cobordant to a Poincare object (M', q') with $\pi_i M' \simeq 0$ for $i < k$.*

The Poincare object (M', q') of Proposition 7 is k-connective. Moreover, since it is Poincare, we have a canonical equivalence $M' \simeq \Sigma^{2k}\mathbb{D}M'$, so that $\mathbb{D}M'$ is $(-k)$-connective: that is, M' has projective amplitude $\leq k$. Applying Variant 2, we obtain the following:

Proposition 5. *Let R be a connective A_∞-ring with involution. Then every Poincare object (M, q) of $(\text{LMod}_R^{\text{fp}}, \Sigma^{2k} Q^{\text{quad}})$ is cobordant to a Poincare object (M', q'), where M' is a direct summand of $\Sigma^k R^n$ for some n.*

In other words, every class in $L_{-2k}(\text{LMod}_R^{\text{fp}}, \Sigma^{2k} Q^{\text{quad}})$ can be represented by a Poincare object (M, q) which is concentrated in the "middle dimension" k.

Remark 6. Exactly the same analysis applies if we replace $\mathrm{LMod}_R^{\mathrm{fp}}$ by $\mathrm{LMod}_R^{\mathrm{perf}}$. Note that it is important that we are working with quadratic L-theory rather than symmetric L-theory.

We now carry out the analogous discussion for L-groups in odd degrees. Let (M, q) be a quadratic object of $(\mathrm{LMod}_R^{\mathrm{fp}}, \Sigma^{2k+1} Q^{\mathrm{quad}})$. Note that if $M = \Sigma^m R$ for $m \leq k$, then $\Sigma^{2k+1} Q^{\mathrm{quad}}(M) \simeq (\Sigma^{2k+1-2m} R)_{h\Sigma_2}$ is connected for $k \leq m$. Consequently, if (M, q) is a general Poincare object we can always do surgery on any class $\eta \in \pi_m M$ for $m \leq k$, to produce a new quadratic object (M_α, q_α). Let us see what effect this surgery has on the homotopy groups of M.

Choose a class $\eta \in \pi_m M$, so that the quadratic form q on M determines a map

$$\beta : M \to \Sigma^{2k+1} \mathbb{D}(M) \to \Sigma^{2k+1} \mathbb{D}(\Sigma^m R) \simeq \Sigma^{2k+1-m} R.$$

We therefore have a long exact sequence of homotopy groups

$$\pi_{i+1} \Sigma^{2k+1-m} R \to \pi_i \operatorname{fib}(\beta) \to \pi_i M \to \pi_i \Sigma^{2k+1-m} R.$$

Consequently, $\pi_i \operatorname{fib}(\beta) \simeq \pi_i M$ for $i < 2k - m$. In particular, if $m < k$, then the homotopy groups of $\operatorname{fib}(\beta)$ agree with the homotopy groups of M below m. Let us assume that $m < k$. We have a fiber sequence

$$\Sigma^m R \to \operatorname{fib}(\beta) \to M_\alpha,$$

so $\pi_i M_\alpha \simeq \pi_i \operatorname{fib}(\beta) \simeq \pi_i M$ for $i < m$, while $\pi_m M_\alpha$ is obtained from $\pi_m \operatorname{fib}(\beta) \simeq \pi_m M$ by killing the class η.

Arguing as before, we obtain the following result:

Proposition 7. *Let (M, q) be a Poincare object of $(\mathrm{LMod}_R^{\mathrm{fp}}, \Sigma^{2k+1} Q^{\mathrm{quad}})$. Then (M, q) is cobordant to a Poincare object (M', q') with $\pi_i M' \simeq 0$ for $i < k$.*

Unfortunately, this conclusion is not quite as strong. Since M' is Poincare, it is isomorphic to $\Sigma^{2k+1} \mathbb{D}(M')$, so that $\mathbb{D}(M')$ is $(-k-1)$-connective. That is, we know that M' is k-connective and has projective amplitude $\leq k+1$. This is not enough to conclude that M' is the suspension of a projective R-module: rather, we can ensure that it is concentrated in two degrees (in a sense which we will take up in the next lecture).

Remark 8. If (M, q) is a quadratic object of $(\mathrm{LMod}_R^{\mathrm{fp}}, \Sigma^{2k+1}, Q^{\mathrm{quad}})$, we can always do surgery on a class $\eta \in \pi_k M$. The problem is that this surgery does not always simplify matters: it has the effect of killing the class η, but might introduce new elements of π_m.