Algebraic Surgery (Lecture 11)

February 17, 2011

Before getting to the main topic of this lecture, let us pick up a few loose ends from the previous lecture. Recall that for every A_∞-ring R with involution, we defined stable ∞-categories $\mathrm{LMod}_R^{\mathrm{perf}}$ and $\mathrm{LMod}_R^{\mathrm{fp}}$ which are equipped with quadratic functors Q and Q^{symm}. We ask the following question: how general are these examples, among all pairs $(\mathcal{C}, Q)$ where $\mathcal{C}$ is a stable ∞-category and Q a nondegenerate quadratic functor on $\mathcal{C}$?

- Let $\mathcal{C}$ be any stable ∞-category containing an object X. Then the spectrum $\mathrm{Mor}_{\mathcal{C}}(X, X)$ is an A_∞-ring spectrum R. Moreover, the construction $M \mapsto X \wedge_R M$ determines a fully faithful embedding $\mathrm{LMod}_R^{\mathrm{fp}} \to \mathcal{C}$, carrying R to X. The essential image of this functor is the smallest stable subcategory of $\mathcal{C}$ containing X. If $\mathcal{C}$ is idempotent complete, then this functor extends to a fully faithful embedding $\mathrm{LMod}_R^{\mathrm{perf}} \to \mathcal{C}$.

- Suppose now that $\mathcal{C}$ is equipped with a symmetric bilinear functor B, and let $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ be given by the formula $Q(X) = B(X, X)^{h\Sigma_2}$. Let us assume that B is nondegenerate, and denote the associated duality functor by $\mathbb{D}$. The functor $\mathbb{D}$ is a contravariant equivalence from $\mathcal{C}$ to itself. Consequently, for any object $X \in \mathcal{C}$ we have a canonical equivalence of A_∞-rings $\mathrm{Mor}_{\mathcal{C}}(X, X) \simeq \mathrm{Mor}_{\mathcal{C}}(\mathbb{D}(X), \mathbb{D}(X))^{op}$. If (X, q) is a Poincare object of $\mathcal{C}$, then this equivalence determines an involution σ on the A_∞-ring $R = \mathrm{Mor}_{\mathcal{C}}(X, X)$.

We can summarize the above discussion as follows: a pair $(\mathcal{C}, Q)$ is of the form $(\mathrm{LMod}_R^{\mathrm{fp}}, Q\ \ )$ if and only if $Q(X) \simeq B(X, X)^{h\Sigma_2}$ for $X \in \mathcal{C}$, and there exists a Poincare object (M, q) in $\mathcal{C}$ such that M generates $\mathcal{C}$ as a stable ∞-category.

Example 1. Let R be an associative ring, and let $\mathcal{D}^{\mathrm{fp}}(R)$ be the ∞-category of bounded chain complexes of finitely generated free modules. We can regard R as an object of $\mathcal{D}^{\mathrm{fp}}(R)$. Let $A = \mathrm{Mor}(R, R)$. The homotopy groups of A are then given by the formula

$$\pi_i A = \mathrm{Ext}_R^{-i}(R, R) = \begin{cases} R & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

In other words, we can identify A with the discrete A_∞-ring corresponding to R. Following the above outline, we obtain a fully faithful embedding $\mathrm{LMod}_A^{\mathrm{fp}} \to \mathcal{D}^{\mathrm{fp}}(R)$. Since $\mathcal{D}^{\mathrm{fp}}(R)$ is generated by R (under taking fibers and cofibers), we see that this fully faithful embedding is an equivalence.

Our goal in the next few lectures is to obtain a *concrete* description of the quadratic L-theory for a ring R with involution (and, more generally, for an A_∞-ring with involution). The obstacle we have to overcome is this: by definition, elements of $L_0\ \ (R)$ are represented by arbitrary finite complexes of free R-modules $P_\bullet$, equipped with a quadratic form represented by a cycle q in $\mathrm{Hom}(P_\bullet \otimes P_\bullet, R)_{h\Sigma_2}$. We saw in the last lecture that there is a concrete description of what it means to give have a cycle, at least in the special case where R is a commutative ring with trivial involution and $P_\bullet$ is concentrated in a single degree. In this lecture, we describe a mechanism which can be used to show that an arbitrary Poincare object $(P_\bullet, q)$ is cobordant (and therefore represents the same L-theory class) to a Poincare object whose underlying chain complex is concentrated in a single degree. The cobordism itself will be constructed using the method of *surgery*.

Let us begin in a general setting. Let $\mathcal{C}$ be a stable ∞-category, $Q : \mathcal{C}^{op} \to \mathrm{Sp}$ a nondegenerate quadratic functor with polarization B, and $\mathbb{D}$ the associated duality functor. Suppose we are given a fiber sequence

$$X' \stackrel{\alpha}{\to} X \to X/X'$$

in $\mathcal{C}$. Let $q \in \Omega^\infty Q(X)$, and suppose that we are given a nullhomotopy of $q|X' \in \Omega^\infty Q(X')$. We have seen that this is generally not enough information to allow us to descend q to a point of $\Omega^\infty Q(X/X')$, because q may have nontrivial image $b \in \Omega^\infty B(X', X)$. Note however that q does have trivial image in $\Omega^\infty B(X', X')$. In other words, the composite map

$$X' \stackrel{\alpha}{\to} X \to \mathbb{D}(X) \stackrel{\mathbb{D}(\alpha)}{\to} \mathbb{D}(X')$$

is canonically nullhomotopic. We therefore obtain a triangle

$$X' \stackrel{\alpha}{\to} X \stackrel{\beta}{\to} \mathbb{D}(X)$$

in the stable ∞-category $\mathcal{C}$. In general, this triangle is not a fiber sequence. Its failure to be a fiber sequence can be measured by taking *homology*: that is, by extracting the object

$$\mathrm{cofib}(X \to \mathrm{fib}(\beta)) \simeq \mathrm{fib}(\mathrm{cofib}(\alpha) \to \mathbb{D}(X))$$

of $\mathcal{C}$ (which vanishes if and only if the sequence above is a fiber sequence). Let us denote this homology object by X_α. This is abusive: it depends not only on α, but on a choice of nullhomotopy of $q|X'$.

Let us write $X_\alpha = \mathrm{fib}(\beta)/X$. We have seen that there is a fiber sequence

$$Q(X_\alpha) \to Q(\mathrm{fib}(\beta)) \to Q(X') \times_{B(X',X')} B(\mathrm{fib}(\beta), X').$$

The point q determines a point of $\Omega^\infty B(X, X')$, classifying the map $\beta : X \to \mathbb{D}X'$. By construction, this map is canonically nullhomotopy after composition with the map $\mathrm{fib}(\beta) \to X$. Consequently, the restriction $q|\,\mathrm{fib}(\beta)$ has trivial image in $\Omega^\infty(Q(X') \times_{B(X',X')} B(\mathrm{fib}(\beta), X'))$, and therefore lifts to a point $q_\alpha \in \Omega^\infty Q(X_\alpha)$. We may therefore view (X_α, q_α) as another quadratic object of $(\mathcal{C}, Q)$. We say that (X_α, q_α) is obtained from (X, q) via *surgery on α*.

The point q_α determines a map from X_α to its dual. This map can be described more explicitly as follows. Note that if we are given a triangle

$$Y' \to Y \to Y'',$$

in $\mathcal{C}$, then we can dualize to obtain a new triangle

$$\mathbb{D}(Y'') \to \mathbb{D}(Y) \to \mathbb{D}(Y')$$

in $\mathcal{C}$. The process of extracting homology is self-dual (rather, the two descriptions of homology given above are dual to one another). The map $X_\alpha \to \mathbb{D}X_\alpha$ is given by the map on homology induced by a map of triangles

$$\begin{array}{ccccc} X' & \longrightarrow & X & \longrightarrow & \mathbb{D}(X') \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{D}^2(X') & \longrightarrow & \mathbb{D}(X) & \longrightarrow & \mathbb{D}(X'). \end{array}$$

Here the outer maps vertical maps are isomorphisms and the middle map is induced by q. Consequently, the cofiber of the map $X_\alpha \to \mathbb{D}(X_\alpha)$ is given by the homology of the triangle

$$0 \to \mathrm{cofib}(X \to \mathbb{D}(X)) \to 0,$$

which is the same as the cofiber of the map $X \to \mathbb{D}(X)$. In particular, one cofiber vanishes if and only if the other does. We have proven:

Proposition 2. *Let (X, q) be a Poincare object of $\mathcal{C}$. Suppose we are given a map $\alpha : X' \to X$ and a nullhomotopy of $q|X'$, and let (X_α, q_α) be obtained by surgery along α. Then (X_α, q_α) is also a Poincare object of $\mathcal{C}$.*

In fact, we can say more. By construction, q_α and q have the same restriction to $L = \mathrm{fib}(X \to \mathbb{D}(X'))$. The identification of these restrictions determines a map

$$X' \simeq \mathrm{fib}(L \to X_\alpha) \to \mathbb{D}\,\mathrm{cofib}(L \to X) = \mathbb{D}(\mathbb{D}(X')) \simeq X'.$$

Unwinding the definitions, one shows that this map is the identity up to a sign. Consequently, the Poincare object (X, q) and (X_α, q_α) are cobordant, and determine the same element of the abelian group $L_0(\mathcal{C}, Q)$.

Remark 3. With some additional effort, one can show that *all* cobordisms arise via this construction. That is, every Poincare object cobordant to (X, q) has the form (X_α, q_α), for some map $\alpha : X' \to X$ and some nullhomotopy of $q|X'$.