L-Theory of Rings and Ring Spectra (Lecture 10)

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Let R be an associative ring. Recall that we earlier introduced the ∞-category $\mathcal{D}^{\mathrm{perf}}(R)$ whose objects can be identified with bounded complexes of finite projective R-modules.

Definition 1. An *involution* on R is a map $\sigma : R \to R$ satisfying the following conditions:

- $\sigma(a+b) = \sigma(a) + \sigma(b)$
- $\sigma(ab) = \sigma(b)\sigma(a)$
- $\sigma\sigma(a) = a$.

If the ring R is equipped with an involution, then any left R-module M can be regarded as a right R-module, via the formula

$$xa = \sigma(a)x.$$

If M is a left R-module, then the R-linear dual $\mathrm{Hom}_R(M, R)$ has the structure of a right R-module. If R is equipped with an involution σ, we can use σ to regard $\mathrm{Hom}_R(M, R)$ as a left module again. Concretely, the left R-module structure is given by the formula

$$(a\lambda)(x) = \lambda(x)\sigma(a)$$

for $a \in R$, $x \in M$, and $\lambda \in \mathrm{Hom}_R(M, R)$.

Let $P_\bullet$ be a bounded chain complex of finitely generated projective left R-modules. We let $\mathbb{D}(P_\bullet)$ denote the chain complex obtained by applying R-linear duality termwise. Using the involution σ on R, we can regard $\mathbb{D}(P_\bullet)$ is also a bounded chain complex of finitely generated projective left R-modules. The construction

$$P_\bullet \mapsto \mathbb{D}(P_\bullet)$$

determines a (contravariant) equivalence of the ∞-category $\mathcal{D}^{\mathrm{perf}}(R)$ with itself. Let B denote the bilinear functor $\mathcal{D}^{\mathrm{perf}}(R)^{op} \times \mathcal{D}^{\mathrm{perf}}(R)^{op} \to \mathrm{Sp}$ given by the formula $B(P_\bullet, Q_\bullet) = \mathrm{Mor}_{\mathcal{D}^{\mathrm{perf}}(R)}(P_\bullet, \mathbb{D}(Q_\bullet))$. The condition $\sigma^2 = \mathrm{id}$ implies that the bilinear functor B is symmetric.

Let us now describe a generalization of the above construction. The ∞-category Sp of spectra is *symmetric monoidal*: that is, there is an operation $\wedge : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$, called the *smash product*, which is commutative and associative up to coherent homotopy. It therefore makes sense to consider associative algebra objects of Sp: that is, spectra R equipped with a multiplication map

$$R \wedge R \to R$$

which are associative (and unital) up to coherent homotopy. We will refer to such an algebra as an *A_∞-ring.*

If R is an A_∞-ring, we can define an ∞-category LMod_R of *left R-module spectra.* The objects of LMod_R are spectra M equipped with an action map

$$R \wedge M \to M$$

satisfying the usual transitivity property, up to coherent homotopy.

Let $\mathrm{LMod}_R^{\mathrm{fp}}$ denote the smallest stable subcategory of LMod_R which contains the R-module R. Let $\mathrm{LMod}_R^{\mathrm{perf}}$ be the smallest subcategory of LMod_R which contains R and is closed under the formation of direct summands. We say that a left R-module M is *finitely presented* if it belongs to $\mathrm{LMod}_R^{\mathrm{fp}}$, and *perfect* if it belongs to $\mathrm{LMod}_R^{\mathrm{perf}}$.

We say that an A_∞-ring spectrum R is *discrete* if $\pi_i R \simeq 0$ for $i \neq 0$. In this case, R is determined (up to canonical homotopy equivalence) by $\pi_0 R$, which is an ordinary associative ring. Moreover, there is a canonical equivalence of ∞-categories

$$\mathrm{LMod}_R^{\mathrm{perf}} \simeq \mathcal{D}^{\mathrm{perf}}(\pi_0 R) :$$

in other words, we can identify (perfect) R-module spectra with (perfect) chain complexes of ordinary modules over the ordinary associative ring $\pi_0 R$.

The collection of all A_∞-rings is organized into an ∞-category. This ∞-category is acted on by the group Σ_2, where the nontrivial element of Σ_2 sends each A_∞-ring R to the same spectrum equipped with the opposite multiplication; we will denote this A_∞-ring by R^{op}. We can identify left R-module spectra with right R^{op}-module spectra. In particular, if M is a left R-module spectrum, then $\mathrm{Mor}(M, R)$ admits a right R-module structure, and so has the structure of a left module over R^{op}. This construction determines a contravariant equivalence of $\mathrm{LMod}_R^{\mathrm{perf}}$ with $\mathrm{LMod}_{R^{op}}^{\mathrm{perf}}$, which restricts to a contravariant equivalence of $\mathrm{LMod}_R^{\mathrm{fp}}$ with $\mathrm{LMod}_{R^{op}}^{\mathrm{fp}}$.

By an *A_∞-ring with involution*, we will mean a homotopy fixed point for the action of Σ_2 on the ∞-category of A_∞-rings. If R is an A_∞-ring with involution, then the construction $M \mapsto \mathrm{Mor}(M, R)$ determines a duality equivalence

$$\mathbb{D} : \mathrm{LMod}_R^{\mathrm{perf},op} \to \mathrm{LMod}_R^{\mathrm{perf}} .$$

This is classified by a symmetric bilinear functor B on $\mathrm{LMod}_R^{\mathrm{perf}}$. Note that this functor carries $\mathrm{LMod}_R^{\mathrm{fp}}$ to itself.

Remark 2. The class of stable ∞-categories and symmetric bilinear functors constructed above is quite general.

- Let $\mathcal{C}$ be any stable ∞-category containing an object X. Then the spectrum $\mathrm{Mor}_{\mathcal{C}}(X, X)$ is an A_∞-ring spectrum R. Moreover, the construction $M \mapsto X \wedge_R M$ determines a fully faithful embedding $\mathrm{LMod}_R^{\mathrm{fp}} \to \mathcal{C}$, carrying R to X. The essential image of this functor is the smallest stable subcategory of $\mathcal{C}$ containing X. If $\mathcal{C}$ is idempotent complete, then this functor extends to a map $\mathrm{LMod}_R^{\mathrm{perf}} \to \mathcal{C}$.

- Let R and R' be A_∞-rings. Let

  $$B : (\mathrm{LMod}_R^{\mathrm{fp}})^{op} \times (\mathrm{LMod}_{R'}^{\mathrm{fp}})^{op} \to \mathrm{Sp}$$

  be a bilinear functor. Regard R and R' as left modules over themselves. Then R is a right R-module in $\mathrm{LMod}_R^{\mathrm{fp}}$, and similarly for R'. Since B is contravariant, we deduce that $B(R, R')$ is a spectrum with commuting left actions of R and R'; this endows $B(R, R')$ with the structure of a left module over $R \wedge R'$. Let us denote this module by P. We can recover B from P: it is given by

  $$B(M, N) = \mathrm{Mor}_{\mathrm{LMod}_{R \wedge R'}}(M \wedge N, P).$$

- Let R, R', and P be as above, and suppose we are given a point $\eta \in \Omega^\infty P$, corresponding to a map of spectra $S \to P$ where S is the sphere spectrum. Then η determines a map of left R-modules $u_\eta : R \to P$. Suppose that u_η is an isomorphism. Then u_η endows R with the structure of a left R'-module, which commutes with the left R-action of R on itself. All endomorphisms of R as a left R-module are given by the right action of R on itself. Consequently, the left action of R' on P is encoded by a map of A_∞-rings $\sigma : R' \to R^{op}$.

- Now suppose that $R = R'$, and let B be the bilinear functor determined by a left $R \wedge R$-module P. Promoting B to a *symmetric* bilinear functor is equivalent to giving an action of the symmetric group Σ_2 on P, which permutes the two R-actions on P. Suppose that, in addition, we have a point $\overline{\eta} \in \Omega^\infty P^{h\Sigma_2}$ which satisfies the condition above (that is, $\overline{\eta}$ induces an isomorphism $R \to P$). Then $\overline{\eta}$ determines a map $\sigma : R \to R^{op}$, as above. Using the fact that $\overline{\eta}$ is a Σ_2-homotopy fixed point, we see that σ is an involution on R.

Definition 3. Let R be an A_∞-ring with involution σ. We let

$$Q^{\mathrm{s}}_\sigma : (\mathrm{LMod}^{\mathrm{fp}}_R)^{op} \to \mathrm{Sp}$$

$$Q^{\mathrm{q}}_\sigma : (\mathrm{LMod}^{\mathrm{fp}}_R)^{op} \to \mathrm{Sp}$$

be the quadratic functors given by

$$Q^{\mathrm{s}}_\sigma(M) = B(M, M)^{h\Sigma_2} \qquad Q^{\mathrm{q}}_\sigma(M) = B(M, M)_{h\Sigma_2}$$

For every integer n, we let

$$L^{\mathrm{s}}_n(R) = L_0(\mathrm{LMod}^{\mathrm{fp}}_R, \Omega^n Q^{\mathrm{s}}_\sigma) \qquad L^{\mathrm{q}}_n(R) = L_0(\mathrm{LMod}^{\mathrm{fp}}_R, \Omega^n Q^{\mathrm{q}}_\sigma).$$

We will refer to the groups $L^{\mathrm{s}}_*(R)$ as the *symmetric L-groups of* R, and $L^{\mathrm{q}}_*(R)$ as the *quadratic L-groups of* R. Note that these groups depend not only on the A_∞-ring R, but also on the involution σ.

Warning 4. This notation is not standard.

Variant 5. Let R be an associative ring with involution. Then we can choose a discrete A_∞-ring $\overline{R}$ with involution, such that $R \simeq \pi_0 \overline{R}$. (Concretely, $\overline{R}$ is given by the *Eilenberg-MacLane spectrum* HR associated to R.) We let $L^{\mathrm{q}}_*(R) = L^{\mathrm{q}}_*(\overline{R})$ and $L^{\mathrm{s}}_*(R) = L^{\mathrm{s}}_*(\overline{R})$.

Variant 6. In the above definition, we can replace $\mathrm{LMod}^{\mathrm{fp}}_R$ with the larger ∞-category $\mathrm{LMod}^{\mathrm{perf}}_R$ of perfect R-modules. Sometimes, this makes no difference (for example, if $R = \mathbf{Z}$), but in general it leads to different L-groups. These are sometimes called the *projective* (symmetric and quadratic) L-groups of R.

Remark 7. Let R be an associative ring with involution σ and let M be a free R-module of finite rank. Then $B(M, M)$ can be identified with the abelian group $\mathrm{Hom}_R(M, \mathrm{Hom}_R(M, R))$ of bilinear forms on M: that is, maps $b : M \times M \to R$ which are additive in each varaible and satisfy

$$b(ax, a'x') = ab(x, x')\sigma(a').$$

We say that b is *symmetric* if $b(x, y) = \sigma b(y, x)$. Promoting M to a quadratic object of $(\mathcal{D}^{\mathrm{fp}}(R), Q^{\mathrm{s}}_\sigma)$ is equivalent to choosing a symmetric bilinear form b on M. In this case, the pair (M, b) is a Poincare object of $(\mathcal{D}^{\mathrm{fp}}(R), Q^{\mathrm{s}}_\sigma)$ if and only if b is nondegenerate (that is, b induces an isomorphism $M \to \mathrm{Hom}_R(M, R)$). We can summarize our analysis as follows: the symmetric L-theory of an associative ring R with involution is closely related to the theory of R-modules equipped with symmetric bilinear forms. In particular, every symmetric bilinear form on R^n determines a class in $L^{\mathrm{s}}_0(R)$.

Remark 8. Let R be a commutative ring, which we regard as an associative ring with involution where the involution is given by id_R. Let M be a free R-module of finite rank.

A *quadratic form* $q : M \to R$ is a map satisfying the following conditions:

(i) $q(ax) = a^2 q(x)$

(ii) $q(x + y) - q(x) - q(y)$ is a bilinear map $M \times M \to R$.

Every bilinear form $b : M \times M \to R$ determines a quadratic form $q : M \to R$ by the formula $q(x) = b(x, x)$. Moreover, every quadratic form arises in this way: if we choose a basis $x_1, \ldots, x_n$ for M over R, then we can define a bilinear form b by the formula

$$b(x_i, x_j) = \begin{cases} q(x_i) & \text{if } i = j \\ q(x_i + x_j) - q(x_i) - q(x_j) & \text{if } i < j \\ 0 & \text{if } i > j. \end{cases}$$

Note that if $\epsilon : M \times M \to R$ is any other bilinear form and we set $b'(x, y) = b(x, y) + \epsilon(x, y) - \epsilon(y, x)$, then b' and b determine the same quadratic form on M. Conversely, suppose that b' and b determine the same quadratic form on M, and define a bilinear form ϵ by the formula

$$\epsilon(x_i, x_j) = \begin{cases} b'(x_i, x_j) - b(x_i, x_j) & \text{if } i < j \\ 0 & \text{if } i \geq j. \end{cases}$$

A simple calculation gives $b'(x, y) = b(x, y) + \epsilon(x, y) - \epsilon(y, x)$. We can summarize this discussion as follows: the abelian group of quadratic forms on M is given by the 0th homology of the group Σ_2 acting on the abelian group of all bilinear forms on M. This homology group can be identified with $\pi_0 B(M, M)_{h\Sigma_2}$. Consequently, up to homotopy, equipping M with the structure of a quadratic object of $(\mathcal{D}^{\text{fp}}(R), Q^{\text{q}}_{\sigma})$ is equivalent to choosing a quadratic form q on M. The pair (M, q) is a Poincare object if and only if the polarization $q(x + y) - q(x) - q(y)$ is a nondegenerate symmetric bilinear form on M.

We can summarize the above discussion as follows: the quadratic L-theory of a commutative ring R (with the identity involution) is closely related to the theory of quadratic forms on R-modules. In particular, every nondegenerate quadratic form on R^n determines a class in $L^{\text{q}}_0(R)$.

Remark 9. Let R be an A_∞-ring with involution. The norm map

$$B(M, M)_{h\Sigma_2} \to B(M, M)^{h\Sigma_2}$$

determines a map of quadratic functors $Q^{\text{q}}_{\sigma} \to Q^{\text{s}}_{\sigma}$, which induces an isomorphism after polarization. This construction determines maps of L-groups

$$L^{\text{q}}_*(R) \to L^{\text{s}}_*(R).$$

These maps are isomorphisms if 2 is invertible in $\pi_0 R$, but not in general. For example, these groups are different in the case $R = \mathbf{Z}$.