The Local Calculation: Passage to Germs (Lecture 26)

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Let k be an algebraically closed field, let X be an algebraic curve over k, and let G be a group scheme over X. In this lecture, we will assume for simplicity that G is split reductive.

Suppose we are given an affine scheme $Y = \operatorname{Spec} R$ equipped with a map $Y \to X^T$, which we can identify with a map of sets $\nu : T \to X(R)$. Let $D = |\nu| \subseteq X_R$ denote the divisor determined by ν. For every R-algebra A, we let $D_A = D \times_{\operatorname{Spec} R} \operatorname{Spec} A$ denote the inverse image of D in X_A.

Our goal in this lecture is to sketch the proof of the following:

Theorem 1. *The canonical map* $\theta : \operatorname{Ran}^{\dagger}_G(X)^T \times_{X^T} Y \to \operatorname{Ran}^G_{\text{germ}}(X)^T \times_{X^T} Y$ *induces an isomorphism on* $\mathbf{Z}_\ell$*-homology.*

We will prove Theorem 1 by factoring θ as a composition of maps

$$\operatorname{Ran}^{\dagger}_G(X)^T \times_{X^T} Y \simeq \mathcal{C}_0 \to \mathcal{C}_1 \to \mathcal{C}_2 \to \mathcal{C}_3 \to \mathcal{C}_4 \to \mathcal{C}_5 \to \mathcal{C}_6 = \operatorname{Ran}^G_{\text{germ}}(X)_T \times_{X^T} Y,$$

each of which induces an isomorphism on $\mathbf{Z}_\ell$-homology. To avoid an overly cumbersome exposition, we will describe each of these prestacks merely by specifying its objects.

Let us begin with $\mathcal{C}_0 = \operatorname{Ran}^{+}_G(X)_T \times_{X^T} Y$. Unwinding the definitions, we can identify the objects of $\mathcal{C}_0$ with tuples $(A, S, K_-, K_+, \mu, \mathcal{P}, \gamma)$ where A is a finitely generated R-algebra, S is a nonempty finite set, $K_- \subseteq K_+ \subseteq S$, $\mu : S \to X(A)$ is a map of sets, $\mathcal{P}$ is a G-bundle on $X_A - |\mu(K_-)|$ (which is extendible to a G-bundle on X_A), and γ is a trivialization of $\mathcal{P}$ on $X_A - |\mu(S)|$. Furthermore, we require that $|\mu(K_+)|$ does not intersect the divisor D_A.

Note that if $\mu : S \to X(A)$ has been specified, then there is a largest subset $K_{\max} \subseteq S$ such that $|\mu(K_{\max})| \cap D_A = \emptyset$. We will say that a tuple $(A, S, K_-, K_+, \mu, \mathcal{P}, \gamma)$ is *maximal* if $K_+ = K_{\max}$. The collection of maximal tuples span a full subcategory $\mathcal{C}_1 \subseteq \mathcal{C}_0$. The inclusion $\mathcal{C}_1 \hookrightarrow \mathcal{C}_0$ admits a left adjoint, given on objects by

$$(A, S, K_-, K_+, \mu, \mathcal{P}, \gamma) \mapsto (A, S, K_-, K_{\max}, \mu, \mathcal{P}, \gamma).$$

It follows formally that this left adjoint induces an isomorphism on $\mathbf{Z}_\ell$-homology.

The prestack $\mathcal{C}_2$ has objects $(A, S, I, \mu, \mathcal{P}, \gamma)$, where A is a finitely generated R-algebra, S is a nonempty finite set, $\mu : S \to X(A)$ is a map, I is a subset of $\mu(S) \subseteq X(A)$ such that $|I| \cap D_A = \emptyset$, $\mathcal{P}$ is a G-bundle on $X_A - |I|$ which can be extended to a G-bundle on X_A, and γ is a trivialization of $\mathcal{P}$ on $X_A - |\mu(S)|$. There is a forgetful functor $\mathcal{C}_1 \to \mathcal{C}_2$, given on objects by the formula

$$(A, S, K_-, K_{\max}, \mu, \mathcal{P}, \gamma) \mapsto (A, S, \mu, \mu(K_-), \mathcal{P}, \gamma).$$

This functor also admits a left adjoint, given by

$$(A, S, \mu, I, \mathcal{P}, \gamma) \mapsto (A, S, \mu^{-1}(I), S_{\max}, \mu, \mathcal{P}, \gamma),$$

and therefore induces an isomorphism on $\mathbf{Z}_\ell$-homology.

We let $\mathcal{C}_3$ denote the category whose objects are tuples $(A, I, J, \mathcal{P}, \gamma)$, where A is a finitely generated R-algebra, I and J are finite subsets of $X(A)$ with $I \subseteq J$, $\mathcal{P}$ is a G-bundle on $X_A - |I|$ which can be extended to a G-bundle on X_A, γ is a trivialization of $\mathcal{P}$ on $X_A - |J|$, and $|I|$ does not intersection D_A. Here the morphisms in $\mathcal{C}_3$ allow the finite set J to become larger. We have a forgetful functor $\mathcal{C}_2 \to \mathcal{C}_3$, given on objects by the formula

$$(A, S, \mu, I, \mathcal{P}, \gamma) \mapsto (A, I, \mu(S), \mathcal{P}, \gamma).$$

This functor fits into a pullback diagram of prestacks

$$\begin{array}{ccc} \mathcal{C}_2 & \longrightarrow & \mathcal{C}_3 \\ \downarrow & & \downarrow \\ \mathrm{Ran}(X) & \longrightarrow & \mathrm{Ran}^+(X). \end{array}$$

We saw earlier that the bottom horizontal map is a universal homology equivalence. Since the vertical maps are op-fibrations, it follows that the map $\mathcal{C}_2 \to \mathcal{C}_3$ induces an isomorphism on $\mathbf{Z}_\ell$-homology.

Let $\mathcal{C}_4$ denote the category whose objects are triples $(A, I, \mathcal{P})$, where A is a finitely generated R-algebra, I is a finite subset of $X(A)$ such that $|I|$ does not intersection D_A, and $\mathcal{P}$ is a G-bundle on $X_A - |I|$ which can be extended to a G-bundle on X_A. We have an evident forgetful functor $\mathcal{C}_3 \to \mathcal{C}_4$, given on objects by

$$(A, I, J, \mathcal{P}, \gamma) \mapsto (A, I, \mathcal{P}).$$

We claim that this map is a universal homological equivalence. To prove this, fix an object $C = (A, I, \mathcal{P}) \in \mathcal{C}_4$ and let $\overline{\mathcal{P}}$ be a G-bundle on X_A extending $\mathcal{P}$. We wish to show that the fiber product $\mathcal{D} = \mathcal{C}_3 \times_{\mathcal{C}_4} (\mathcal{C}_4)_{C/}$ has homology isomorphic to that of $\operatorname{Spec} A$. The objects of $\mathcal{D}$ can be described roughly as pairs (B, γ), where B is a finitely generated A-algebra and γ is a trivialization of $\operatorname{Spec} B \times_{\operatorname{Spec} A} \overline{\mathcal{P}}$ which is defined away from a finite subset $J \subseteq X(B)$ containing the image of I (this is a slight oversimplification). As such, it is a retract of the prestack $\mathrm{Sect}^+(\mathcal{P})$ of rational sections of $\mathcal{P}$ (see Lecture 12). It will therefore suffice to show that the map $\mathrm{Sect}^+(\mathcal{P}) \to \operatorname{Spec} A$ induces an isomorphism on $\mathbf{Z}_\ell$-homology, which was one formulation of our nonabelian Poincare duality theorem.

If A is a finitely generated R-algebra, let us say that an open subset $U \subseteq X_A$ is *thick* if it has the form $X_A - |I|$, where I is a finite subset of $X(A)$. Let $\mathcal{C}_5$ denote the category whose objects are pairs $(A, \mathcal{P})$, where A is a finitely generated R-algebra and $\mathcal{P}$ is a G-bundle on some thick open subset $U \subseteq X_A$ which contains D, having the additional property that $\mathcal{P}$ can be extended to a G-bundle on X_A. The construction $(A, I, \mathcal{P}) \mapsto (A, \mathcal{P})$ determines a forgetful functor $\mathcal{C}_4 \to \mathcal{C}_5$. Given a thick open set $U \subseteq X_A$, the collection of those finite sets $I \subseteq X(A)$ such that $U \supseteq X_A - |I|$ is filtered. Using this, one can deduce that the functor $\mathcal{C}_4 \to \mathcal{C}_5$ is right cofinal, and therefore induces an isomorphism on $\mathbf{Z}_\ell$-homology.

Let $\mathcal{C}_6 = \mathrm{Ran}^G(X)^T \times_{X^T} Y$. Unwinding the definitions, we can identify objects of $\mathcal{C}_6$ with pairs $(A, \mathcal{P})$, where A is a finitely generated R-algebra and $\mathcal{P}$ is a G-bundle defined over *some* open subset $U \subseteq X_A$ containing D, which can be extended to a G-bundle on X_A. We can identify $\mathcal{C}_5$ with the full subcategory of $\mathcal{C}_6$ spanned by those objects for which the open set U is thick. Earlier, we showed that for any full open subset $U \subseteq X_A$, we can (after passing to an fppf covering of $\operatorname{Spec} A$) arrange that U contains a thick open subset $V \subseteq X_A$. The argument given there also proves that if U contains a divisor D, then we can arrange that V also contains the divisor D. It follows that the inclusion functor $\mathcal{C}_5 \hookrightarrow \mathcal{C}_6$ induces an equivalence after stackification with respect to the fppf topology. Since the construction $Y \mapsto C_*(Y; \mathbf{Z}_\ell)$ satisfies fppf descent, it follows that the inclusion $\mathcal{C}_5 \hookrightarrow \mathcal{C}_6$ induces an isomorphism on $\mathbf{Z}_\ell$-homology. This completes our sketch of the proof of Theorem 1.