The Local Calculation: Outline (Lecture 25)

April 10, 2014

Let k be an algebraically closed field, let X be an algebraic curve over k, and let G be a smooth affine group scheme over X.

In the last lecture, we introduced a family of prestacks $\{\mathrm{Ran}_G^\dagger(X)_S\}_{S \in \mathrm{Fin}^s}$ equipped with maps $\phi_S : \mathrm{Ran}_G^\dagger(X)_S \to \mathrm{Ran}(X)$, and a family of !-sheaves $\mathcal{B}_S$ given informally by the formulae

$$\mathcal{B}_S = [\mathrm{Ran}_G^\dagger(X)_S]_{\mathrm{Ran}(X)} = \phi_{S*}\phi_S^*\omega_{\mathrm{Ran}(X)}.$$

Our goal for the next several lectures is to prove the following:

Proposition 1. *Assume that the generic fiber of G is semisimple and simply connected. Then the canonical maps $\{\mathrm{Ran}_G^\dagger(X)_S \to \mathrm{Ran}^G(X)\}_{S \in \mathrm{Fin}^s}$ induce an equivalence*

$$\mathcal{B} \to \varprojlim_{S \in \mathrm{Fin}^s} \mathcal{B}_S$$

in the ∞-category $\mathrm{Shv}^!(\mathrm{Ran}(X))$.

By its nature, Proposition 1 is "local" on $\mathrm{Ran}(X)$. To prove it, it will suffice to show that for every nonempty finite set T, the underlying map

$$\theta_T : [\mathrm{Ran}^G(X)^T]_{X^T} \to \varprojlim_S [\mathrm{Ran}_G^\dagger(X)_S^T]_{X^T}$$

is an equivalence in $\mathrm{Shv}_\ell(X^T)$, where $\mathrm{Ran}^G(X)^T$ denotes the fiber product $\mathrm{Ran}^G(X) \times_{\mathrm{Ran}(X)} X^T$, and $\mathrm{Ran}_G^\dagger(X)_S^T$ is defined similarly. In fact, we will prove the following stronger assertion:

Proposition 2. *Let T be a nonempty finite set, fixed throughout this lecture. Let Y be a quasi-projective k-scheme equipped with a map $Y \to X^T$. Then the canonical map*

$$\theta_Y : [\mathrm{Ran}^G(X)^T \times_{X^T} Y]_Y \to \varprojlim_S [\mathrm{Ran}_G^\dagger(X)_S^T \times_{X^T} Y]_Y$$

is an equivalence in $\mathrm{Shv}_\ell(Y)$.

The virtue the formulation given in Proposition 2 is that it will allow us to apply a devissage to the scheme Y. Suppose we are given a pullback diagram of k-schemes

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{f} & Y, \end{array}$$

where the horizontal maps are proper. We have seen that this diagram induces a map $[U']_{Y'} = g'_* g'^* \omega_{Y'} \to f^! g_* g^* \omega_Y = f^! [U]_Y$. If the vertical maps are smooth, then the smooth base change theorem implies that this map is invertible: that is, we can identify $[U']_{Y'}$ with $f^![U]_{Y'}$. One can show that this holds more generally for commutative diagrams of prestacks

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{f} & Y, \end{array}$$

provided satisfying one of the following conditions:

(a) The map g exhibits $\mathcal{C}$ as an Artin stack which is smooth over Y (this condition is satisfied by the morphisms $\mathrm{Ran}^G(X)^T \times_{X^T} Y \to Y$).

(b) The prestack $\mathcal{C}$ admits an open immersion into a product $\mathcal{C}_0 \times_{\operatorname{Spec} k} Y$. (This condition is satisfied by the morphisms $\mathrm{Ran}^{\dagger}_G(X)^T_S \times_{X^T} Y \to Y$.)

It follows that for any proper map $f : Y' \to Y$ of X^T-schemes, we can identify $\theta_{Y'}$ with the image of θ_Y under the functor $f^! : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(Y')$.

Remark 3. Suppose that $i : Y' \to Y$ is a closed immersion, with complementary open immersion $j : U \to Y$. For any object $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$, we have a canonical fiber sequence

$$i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F}.$$

In particular, $\mathcal{F} \simeq 0$ if and only if both $i^! \mathcal{F}$ and $j^* \mathcal{F}$ vanish. It follows that θ_Y is an equivalence if and only if $i^!(\theta_Y) \simeq \theta_{Y'}$ and $j^*(\theta_Y) \simeq \theta_U$ are equivalences.

The proof of Proposition 2 will proceed by Noetherian induction on Y. That is, to prove that θ_Y is an equivalence, we may assume without loss of generality that $\theta_{Y'}$ is an equivalence for every closed subscheme $Y' \subsetneq Y$. If Y is non-reduced, we can complete the proof by taking $Y' = Y_{\mathrm{red}}$. Let us assume that Y is nonempty (otherwise, there is nothing to prove). By virtue of Remark 3, it will suffice to prove Proposition 2 after replacing Y by an arbitrary nonempty open subset of Y. We may therefore assume without loss of generality that $Y = \operatorname{Spec} R$ is smooth and affine. In this case, the map $Y \to X^T$ corresponds to a map $\nu : T \to X(R)$.

Remark 4. In class, we will eventually specialize to the case where the group scheme G is split reductive (in which case the proof becomes dramatically simpler). If this condition were not satisfied, it would be convenient at this point to assume in addition that the map $Y \to X^T$ is "transverse" to G: that is, that each of the maps $\nu(t) : \operatorname{Spec} R \to X$ is either constant or has image disjoint from the locus where G is not reductive.

Let us say that an object $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$ is *ℓ-adically complete* if limit of the tower

$$\cdots \to \mathcal{F} \xrightarrow{\ell} \mathcal{F} \xrightarrow{\ell} \mathcal{F}$$

vanishes. Equivalently, $\mathcal{F}$ is ℓ-adically complete if it can be recovered as the limit of the tower

$$\cdots \to (\mathbf{Z}/\ell^3\mathbf{Z}) \otimes_{\mathbf{Z}_\ell} \mathcal{F} \to (\mathbf{Z}/\ell^2\mathbf{Z}) \otimes_{\mathbf{Z}_\ell} \mathcal{F} \to (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}_\ell} \mathcal{F}.$$

Any constructible sheaf is ℓ-adically complete, and the collection of ℓ-adically complete objects of $\mathrm{Shv}_\ell(Y)$ is closed under limits. It follows that for every map of prestacks $\mathcal{C} \to Y$, the sheaf $[\mathcal{C}]_Y \in \mathrm{Shv}_\ell(Y)$ is ℓ-adically complete. In particular, θ_Y is a morphism between ℓ-adically complete objects of $\mathrm{Shv}_\ell(Y)$. Consequently, to prove that θ_Y is an equivalence, it will suffice to show that θ_Y induces an equivalence after tensoring with $\mathbf{Z}/\ell^d\mathbf{Z}$, for every integer $d \geq 0$. In other words, it will suffice to prove the analogue of Proposition 2 after replacing $\mathbf{Z}_\ell$ by $\mathbf{Z}/\ell^d\mathbf{Z}$. Note that we can detect equivalences in $\mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z})$ by passing to global sections over étale Y-schemes V. Replacing Y by V, we are reduced to proving that the canonical map

$$[\mathrm{Ran}^G(X)^T \times_{X^T} Y]_Y(Y) \to \varprojlim_S [\mathrm{Ran}_G^\dagger(X)_S^T \times_{X^T} Y]_Y(Y)$$

is a quasi-isomorphism. Since Y is smooth, the dualizing complex ω_Y agrees with the constant sheaf on Y up to a shift, so that we identify $[\mathcal{C}]_Y(Y)$ with a shift of $C^*(\mathcal{C}; \mathbf{Z}/\ell^d\mathbf{Z})$ for any prestack $\mathcal{C}$ over Y. We are therefore reduced to proving that the canonical map

$$C^*(\mathrm{Ran}^G(X)^T \times_{X^T} Y; \mathbf{Z}/\ell^d\mathbf{Z}) \to \varprojlim_S C^*(\mathrm{Ran}_G^\dagger(X)_S^T \times_{X^T} Y; \mathbf{Z}/\ell^d\mathbf{Z})$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}/\ell^d\mathbf{Z}}$. In fact, we will prove a stronger assertion at the level of homology. For simplicity, let us henceforth assume that the group scheme G is *constant*.

Proposition 5. *Suppose we are given a map* $Y = \mathrm{Spec}\, R \to X^T$*, corresponding to a map* $\nu : T \to X(R)$ *which is in general position. Then the canonical map*

$$\varinjlim_S C_*(\mathrm{Ran}_G^\dagger(X)_S^T \times_{X^T} Y; \mathbf{Z}_\ell) \to C_*(\mathrm{Ran}^G(X)^T \times_{X^T} Y; \mathbf{Z}_\ell)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Remark 6. Proposition 5 can be generalized to the case of a non-constant group scheme G, but the notion of "general position" needs to be slightly modified.

Let us now outline our strategy for proving Proposition 5. First, $\mathrm{Ran}_G^\dagger(X)^T$ denote the prestack obtained by applying the Grothendieck construction to the functor $S \mapsto \mathrm{Ran}_G^\dagger(X)_S^T$. More precisely, $\mathrm{Ran}_G^\dagger(X)^T$ denotes the category whose objects are tuples $(A, S, K_-, K_+, \mu, \nu, \mathcal{P}, \gamma)$ where A is a finitely generated k-algebra, S is a nonempty finite set, K_- and K_+ are subsets of S with $K_- \subseteq K_+$, $\mu : S \to X(A)$ and $\nu : T \to X(A)$ are maps such that $|\mu(K_+)|$ and $|\nu(T)|$ do not intersect, $\mathcal{P}$ is a G-bundle on $X_A - |\mu(K_-)|$ which can be extended to a G-bundle on X_A, and γ is a trivialization of $\mathcal{P}$ over $X_A - |\mu(S)|$. Then we can identify the direct limit $\varinjlim_S C_*(\mathrm{Ran}_G^\dagger(X)_S^T \times_{X^T} Y; \mathbf{Z}_\ell)$ with $C_*(\mathrm{Ran}_G^\dagger(X)_T \times_{X^T} Y; \mathbf{Z}_\ell)$. It will therefore suffice to show that the forgetful functor

$$\mathrm{Ran}_G^\dagger(X)^T \times_{X^T} Y \to \mathrm{Ran}^G(X)^T \times_{X^T} Y$$

induces an isomorphism on homology. To prove this, we will need an auxiliary constructions:

Definition 7. We define a category $\mathrm{Ran}^G_{\mathrm{germ}}(X)^T$ as follows:

- (a) The objects of $\mathrm{Ran}^G_{\mathrm{germ}}(X)^T$ are triples $(A, \nu, \mathcal{P})$ where A is a finitely generated k-algebra, $\nu : T \to X(A)$ is a map, and $\mathcal{P}$ is a G-bundle on X_A.
- (b) A morphism from $(A, \nu, \mathcal{P})$ to $(A', \nu', \mathcal{P}')$ is a k-algebra homomorphism $A \to A'$ such that ν' coincides with the composite map $T \xrightarrow{\nu} X(A) \to X(A')$, together with a germ of G-bundle isomorphisms of $X_{A'} \times_{X_A} \mathcal{P}$ with $\mathcal{P}'$ around the divisor $|\nu'| \subseteq X_{A'}$ (that is, we require an isomorphism which is defined on some open subset of $X_{A'}$ which contains $|\nu'|$).

We have evident forgetful functors

$$\mathrm{Ran}_G^\dagger(X)^T \to \mathrm{Ran}^G_{\mathrm{germ}}(X)^T \to \mathrm{Ran}^G(X)^T.$$

To prove Proposition 5, it will suffice to show that for every map $Y = \mathrm{Spec}\, R \to X^T$, the maps

$$\mathrm{Ran}_G^+(X)^T \times_{X^T} Y \xrightarrow{\rho_0} \mathrm{Ran}^G_{\mathrm{germ}}(X)^T \times_{X^T} Y \xrightarrow{\rho_1} \mathrm{Ran}^G(X)^T \times_{X^T} Y$$

induce isomorphisms on $\mathbf{Z}_\ell$-homology. We will take this up in the next lecture.