Verdier Duality (Lecture 21)

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Let k be an algebraically closed field, ℓ a prime number which is invertible in k, X an algebraic curve over k and G a reductive group scheme over X. The direct image of the constant sheaf along the map $\mathrm{Ran}_G(X) \to \mathrm{Ran}(X)$ can be regarded as a sheaf $\mathcal{A}$ on the Ran space $\mathrm{Ran}(X)$, whose stalk at a point $\mu : S \to X(k)$ having image $\{x_1, \ldots, x_m\} \subseteq X(k)$ is given by

$$\mathcal{A}_\mu \simeq \bigotimes C^*(\mathrm{Gr}_{G,x_i}; \mathbf{Z}_\ell).$$

The main result of the first part of this course is that the cochain complex $C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ can be identified with the global sections of $\mathcal{A}$. Roughly speaking, our next move is to exploit this fact by analyzing the Verdier dual of $\mathcal{A}$ (or, more precisely, the Verdier dual of a "reduced version" of $\mathcal{A}$ which we will discuss in the next two lectures). Since the Ran space $\mathrm{Ran}(X)$ is an infinite-dimensional algebro-geometric object, the classical theory of Verdier duality does not apply directly. In this lecture, we will give an overview of Verdier duality in the setting of topology, and describe how it can be adapted to spaces like $\mathrm{Ran}(X)$.

We begin by reviewing the theory of sheaves. For the remainder of this lecture, let us fix an ∞-category $\mathcal{C}$ which admits small limits and colimits.

Let Y be a topological space and let $\mathcal{U}(Y)$ denote the partially ordered set of all open subsets of Y. Recall that a $\mathcal{C}$-*valued sheaf* on Y is a functor $\mathcal{F} : \mathcal{U}(Y)^{\mathrm{op}} \to \mathcal{C}$ with the following property: for each open cover $\{U_\alpha\}$ of an open set $U \subseteq Y$, the canonical map

$$\mathcal{F}(U) \to \varprojlim_V \mathcal{F}(V)$$

is an equivalence in $\mathcal{C}$, where V ranges over all open subsets of U which are contained in some U_α. The collection of all $\mathcal{C}$-valued sheaves on Y can be organized into an ∞-category, which we will denote by $\mathrm{Shv}_{\mathcal{C}}(Y)$.

There is an evident dual notion of $\mathcal{C}$-*valued cosheaf on* Y: a functor $\mathcal{F} : \mathcal{U}(Y) \to \mathcal{C}$ with the property that for every open cover $\{U_\alpha\}$ of an open set $U \subseteq Y$, the canonical map $\varinjlim_V \mathcal{F}(V) \to \mathcal{F}(U)$ is an equivalence in $\mathcal{C}$ (again the colimit is taken over all open subsets of U which are contained in some U_α). The collection of all $\mathcal{C}$-valued cosheaves on Y can be organized into an ∞-category which we will denote by $\mathrm{cShv}_{\mathcal{C}}(Y)$. Equivalently, we can define the ∞-category $\mathrm{cShv}_{\mathcal{C}}(Y)$ by the formula

$$\mathrm{cShv}_{\mathcal{C}}(Y) = \mathrm{Shv}_{\mathcal{C}^{\mathrm{op}}}(Y)^{\mathrm{op}}.$$

Let us now suppose that $\mathcal{C}$ is *pointed*: that is, there exists an object $0 \in \mathcal{C}$ which is both initial and final. For any morphism $f : C \to C'$ in $\mathcal{C}$, the *fiber* of f is defined to be the pullback $C \times_{C'} 0$. For each object $C \in \mathcal{C}$, we let $\Omega(C)$ denote the fiber of the zero map $0 \to C$ (that is, the fiber product $0 \times_C 0$). The construction $C \mapsto \Omega(C)$ determines a functor $\Omega : \mathcal{C} \to \mathcal{C}$. We say that $\mathcal{C}$ is *stable* if this functor is invertible.

Example 1. For any associative ring R, the ∞-category Mod_R of chain complexes over R is a stable ∞-category; the functor $\Omega : \mathrm{Mod}_R \to \mathrm{Mod}_R$ can be implemented by shifting the grading of a chain complex.

Remark 2. If $\mathcal{C}$ is a stable ∞-category, then its homotopy category $\mathrm{h}\mathcal{C}$ admits the structure of a triangulated category, whose "shift" functor is given by a homotopy inverse of the functor $\Omega : \mathcal{C} \to \mathcal{C}$.

Let us now henceforth assume that we are working with a stable ∞-category $\mathcal{C}$. In this case, there is a closed relationship between sheaves and cosheaves.

Construction 3. Let Y be a Hausdorff topological space and let $\mathcal{F}$ be a $\mathcal{C}$-valued cosheaf on Y. For every closed subset $K \subseteq Y$, we let $\mathcal{F}_K$ denote the fiber of the natural map $\mathcal{F}(Y) \rightarrow \mathcal{F}(Y - K)$. We can think of $\mathcal{F}_K$ as the "space of sections of $\mathcal{F}$ supported on K". Note that $\mathcal{F}_K$ is a *covariant* functor of K.

Given an open set $U \subseteq Y$, we let $\mathcal{F}_c(U)$ denote the direct limit $\varinjlim_{K \subseteq U} \mathcal{F}_K$, where the colimit is taken over all compact subsets $K \subseteq U$. We can think of $\mathcal{F}_c(U)$ as the space of sections of $\mathcal{F}$ which are supported in a compact subset of U. Then $\mathcal{F}_c$ is a covariant functor of U.

Remark 4. For $K \subseteq U$, we can identify $\mathcal{F}_K$ with the fiber of the restriction map $\mathcal{F}(U) \rightarrow \mathcal{F}(U - K)$. It follows that there is a canonical map $\mathcal{F}_c(U) \rightarrow \mathcal{F}(U)$ for every open subset $U \subseteq Y$.

Theorem 5 (Covariant Verdier Duality)**.** *Let $\mathcal{C}$ be a stable ∞-category and let Y be a Hausdorff space. Then the construction $\mathcal{F} \mapsto \mathcal{F}_c$ carries $\mathcal{C}$-valued sheaves on Y to $\mathcal{C}$-valued cosheaves on Y. If Y is locally compact, then the construction $\mathcal{F} \mapsto \mathcal{F}_c$ determines an equivalence of ∞-categories*

$$\mathrm{Shv}_{\mathcal{C}}(Y) \simeq \mathrm{cShv}_{\mathcal{C}}(Y).$$

In other words, if Y is a locally compact space, then the ∞-categories $\mathrm{Shv}_{\mathcal{C}}(Y)$ and $\mathrm{Shv}_{\mathcal{C}^{\mathrm{op}}}(Y)$ are canonically equivalent to the opposites of one another. This equivalence is nontrivial, and involves the topology of Y in an essential way.

Remark 6 (Functoriality)**.** Let $f : Y \rightarrow Z$ be a continuous map of topological spaces. For any ∞-category $\mathcal{C}$, f determines a pushforward functor $f_* : \mathrm{Shv}_{\mathcal{C}}(Y) \rightarrow \mathrm{Shv}_{\mathcal{C}}(Z)$ by the formula $(f_* \mathcal{F})(U) = \mathcal{F}(f^{-1}U)$. Similarly, we have a pushforward operation on cosheaves $f_+ : \mathrm{cShv}_{\mathcal{C}}(Y) \rightarrow \mathrm{cShv}_{\mathcal{C}}(Z)$, given by $(f_+ \mathcal{F})(U) = \mathcal{F}(f^{-1}U)$.

Under mild assumptions on $\mathcal{C}$, the pushforward functor f_* admits a left adjoint f^*. Under similar hypotheses on $\mathcal{C}^{\mathrm{op}}$, we see that the pushforward functor f_+ admits a right adjoint f^+.

Suppose now that $\mathcal{C}$ is stable and that Y and Z are locally compact. In this case, Verdier duality supplies equivalences of ∞-categories

$$\mathrm{Shv}_{\mathcal{C}}(Y) \simeq \mathrm{cShv}_{\mathcal{C}}(Y) \qquad \mathrm{Shv}_{\mathcal{C}}(Z) \simeq \mathrm{cShv}_{\mathcal{C}}(Z).$$

Under these equivalences, we can identify f_+ and f^+ with adjoint functors

$$f_! : \mathrm{Shv}_{\mathcal{C}}(Y) \rightarrow \mathrm{Shv}_{\mathcal{C}}(Z) \qquad f^! : \mathrm{Shv}_{\mathcal{C}}(Z) \rightarrow \mathrm{Shv}_{\mathcal{C}}(Y).$$

Unwinding the definitions, we see that the functor $f_!$ is characterized by the formula $(f_! \mathcal{F})_c(U) = \mathcal{F}_c(f^{-1}U)$. We will refer to $f_!$ as the functor of *direct image with compact supports.*

Example 7. Let R be a commutative ring, and let Mod_R denote the ∞-category of chain complexes over R. Then R-linear duality defines a contravariant functor from Mod_R to itself, which carries colimits to limits. Consequently, every Mod_R-valued cosheaf $\mathcal{G}$ on a space Y determines a Mod_R-valued sheaf $\mathcal{G}^\vee$, given by $\mathcal{G}^\vee(U) = (\mathcal{G}(U))^\vee$.

Let $\mathcal{F}$ be a Mod_R-valued sheaf on a Hausdorff space Y, and let $\mathcal{F}_c$ be the associated cosheaf. We let $\mathbf{D}(\mathcal{F})$ denote the Mod_R-valued sheaf $\mathcal{F}_c^\vee$. We refer to $\mathbf{D}(\mathcal{F})$ as the *Verdier dual* of $\mathcal{F}$. We let ω_Y denote the Verdier dual of the constant sheaf $\underline{R}_Y$ on Y. We refer to ω_Y as the *dualizing sheaf* of Y.

Assuming that Y is locally compact, for any pair of sheaves $\mathcal{F}$ and $\mathcal{G}$ we have an equivalence

$$\mathrm{Map}_{\mathrm{Shv}_{\mathrm{Mod}_R}(Y)}(\mathcal{F}, \mathbf{D}(\mathcal{G})) \simeq \mathrm{Map}_{\mathrm{Shv}_{\mathrm{Mod}_R}(Y)}(\mathcal{G}, \mathbf{D}(\mathcal{F})).$$

In particular, we obtain an equivalence

$$\begin{aligned} \mathcal{F}(Y) &\simeq \mathrm{Map}_{\mathrm{Shv}_{\mathrm{Mod}_R}(Y)}(\underline{R}, \mathbf{D}(\mathcal{F})) \\ &\simeq \mathrm{Map}_{\mathrm{Shv}_{\mathrm{Mod}_R}(Y)}(\mathcal{F}, \mathbf{D}(\underline{R})) \\ &\simeq \mathrm{Map}_{\mathrm{Shv}_{\mathrm{Mod}_R}(Y)}(\mathcal{F}, \omega_R) \end{aligned}$$

of chain complexes over R. Performing a similar calculation over each open subset of R, we see that $\mathbf{D}(\mathcal{F})$ can be identified with sheaf which classifies maps from $\mathcal{F}$ into ω_R.

Let us now consider what Verdier duality can tell us about a topological space Y which is not locally compact, such as the Ran space $\mathrm{Ran}(M)$ of a manifold M. For each integer $n \geq 0$, let $\mathrm{Ran}_{\leq n}(M)$ denote the subset of $\mathrm{Ran}(M)$ consisting of finite subsets $S \subseteq M$ which have cardinality $\leq n$. Each $\mathrm{Ran}_{\leq n}(M)$ is a locally compact topological space (which can be identified with a quotient of M^n): even a compact topological space if M is compact. We will endow $\mathrm{Ran}(M)$ with the direct limit topology (so that a subset $U \subseteq \mathrm{Ran}(M)$ is open if and only if its intersection with each $\mathrm{Ran}_{\leq n}(M)$ is open). Note that this is not quite the same as the topology introduced in Lecture 15; however, this will not play an important role in our discussion.

We refer to [2] for a proof of the following:

Proposition 8. *Let Y be a paracompact topological space which is written as a direct limit of closed subsets*

$$Y_1 \hookrightarrow Y_2 \hookrightarrow Y_3 \hookrightarrow \cdots$$

Then $\mathrm{Shv}_{\mathcal{C}}(Y)$ can be identified with the inverse limit $\varprojlim \mathrm{Shv}_{\mathcal{C}}(Y_n)$.

In other words, giving a $\mathcal{C}$-valued sheaf $\mathcal{F}$ on Y is equivalent to giving a sequence of $\mathcal{C}$-valued sheaves $\mathcal{F}_n \in \mathrm{Shv}_{\mathcal{C}}(Y_n)$, together with equivalences $\mathcal{F}_n \simeq i(n)^* \mathcal{F}_{n+1}$, where $i(n) : Y_n \to Y_{n+1}$ denotes the inclusion map. Applying the same reasoning to the ∞-category $\mathcal{C}^{\mathrm{op}}$, we see that $\mathrm{cShv}_{\mathcal{C}}(Y)$ can be identified with the inverse limit of the tower

$$\cdots \to \mathrm{cShv}_{\mathcal{C}}(Y_3) \stackrel{i(2)^+}{\to} \mathrm{cShv}_{\mathcal{C}}(Y_2) \stackrel{i(1)^+}{\to} \mathrm{cShv}_{\mathcal{C}}(Y_1).$$

Suppose now that each of the spaces Y_i is locally compact. Applying Theorem 5 to each term in this sequence, we obtain an equivalent tower

$$\cdots \to \mathrm{Shv}_{\mathcal{C}}(Y_3) \stackrel{i(2)^!}{\to} \mathrm{Shv}_{\mathcal{C}}(Y_2) \stackrel{i(1)^!}{\to} \mathrm{Shv}_{\mathcal{C}}(Y_1).$$

We will denote the inverse limit of this tower by $\mathrm{Shv}^!_{\mathcal{C}}(Y)$, and refer to it as the *∞-category of $\mathcal{C}$-valued !-sheaves* on Y. More informally, the objects of $\mathrm{Shv}^!_{\mathcal{C}}(Y)$ are sequences of sheaves $\mathcal{F}_n \in \mathrm{Shv}_{\mathcal{C}}(Y_n)$, together with equivalences $\mathcal{F}_n \simeq i(n)^! \mathcal{F}_{n+1}$.

The reasoning above supplies an equivalence of ∞-categories $\mathrm{cShv}_{\mathcal{C}}(Y) \simeq \mathrm{Shv}^!_{\mathcal{C}}(Y)$. Consequently, the construction $\mathcal{F} \mapsto \mathcal{F}_c$ determines a functor

$$\Phi : \mathrm{Shv}_{\mathcal{C}}(Y) \to \mathrm{Shv}^!_{\mathcal{C}}(Y),$$

which can be viewed as a covariant version of Verdier duality for the topological space Y. Note that Φ need not be an equivalence of ∞-categories if Y is not locally compact.

Remark 9. If $\mathcal{F} \in \mathrm{Shv}_{\mathcal{C}}(Y)$, we can roughly think of $\Phi(\mathcal{F})$ as a !-sheaf which remembers information only about those sections of $\mathcal{F}$ which are supported on Y_n for some n.

References

[1] Lurie, J. *Higher Algebra.*

[2] Lurie, J. *Higher Topos Theory.* Princeton University Press, 2009.