Existence of Borel Reductions II (Lecture 15)

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Throughout this lecture, we let k be an algebraically closed field, X an algebraic curve over k, G a smooth affine group scheme over X, G_0 the generic fiber of G, $B_0 \subseteq B$ the scheme-theoretic closure of B_0 in G. Let $\mathcal{P}$ be a G-bundle on X and let $\pi : \mathcal{P}/B \to X$ be the projection map. Our goal is to prove the following result which was needed in the previous lecture:

Theorem 1. *Let $\mathcal{P}$ be a G-bundle on X. Then there exists a section s of the projection map $\pi : \mathcal{P}/B \to X$ such that $\mathrm{H}^1(X; s^* T_\pi) \simeq 0$.*

We will prove Theorem 1 under the assumption that the generic fiber G_0 is split reductive. The statement also holds under the assumption that G_0 is semisimple and simply connected, but requires a more complicated argument.

Let G' be the unique split reductive algebraic group over k such that there is an isomorphism $\alpha : \operatorname{Spec} K_X \times_{\operatorname{Spec} k} G' \simeq G_0$ and let B' be a Borel subgroup of G'. Since all Borel subgroups of G_0 are conjugate, we may assume without loss of generality that the isomorphism α carries $\operatorname{Spec} K_X \times_{\operatorname{Spec} k} B'$ to B_0. We may therefore choose a dense open subset $U \subseteq X$ such that α extends to an isomorphism $G' \times_{\operatorname{Spec} k} U \simeq G \times_X U$ carrying $B' \times_{\operatorname{Spec} k} U$ to $B \times_X U$.

In the last lecture, we showed that $\mathcal{P}$ admits a B-reduction, which we can identify with a section s_0 of the projection map $\pi : \mathcal{P}/B \to X$. Let $\mathcal{Q}$ denote the associated B-bundle on X, so that $\mathcal{Q}$ is trivial at the generic point of X. Shrinking the open set U, we may assume that $\mathcal{Q}|_U$ is trivial. It follows that α determines an isomorphism

$$\beta : G'/B' \times_{\operatorname{Spec} k} U \simeq \mathcal{P}/B \times_X U,$$

which carries the "zero section" of the projection map $G'/B' \times_{\operatorname{Spec} k} U \to U$ to the map $s_0|_U$.

We would now like to extend β to a map

$$\overline{\beta} : G'/B' \times_{\operatorname{Spec} k} X \to \mathcal{P}/B.$$

Unfortunately, such an extension need not exist. However, we can always find such an extension after suitably "blowing-up" the variety $G'/B' \times_{\operatorname{Spec} k} X$.

Construction 2 (Dilation)**.** Let Y be a quasi-projective k-scheme equipped with a smooth map $f : Y \to X$, and let $y \in Y(k)$ be a point having image $x \in X(k)$. Let $\mathcal{I}_y$ denote the ideal sheaf of y in Y. Let $\mathcal{A}_y$ denote the direct limit

$$\varinjlim \mathcal{I}_y^m \otimes_{\mathcal{O}_Y} f^* \mathcal{O}_X(mx).$$

Then $\mathcal{A}_y$ is a quasi-coherent sheaf of algebras on Y, which determines a map of affine schemes $D_y(Y) \to Y$. We will refer to $D_y(Y)$ as the *dilitation of* Y at the point y.

Remark 3. We can describe $D_y(Y)$ as the scheme obtained by first blowing up Y at the point y, and then removing the closed subscheme obtained by blowing up $Y \times_X \{x\}$ at the point y.

Remark 4. Suppose that Y is smooth over X. Then $D_y(Y)$ is also smooth over X. Moreover, if $g : D_y(Y) \to Y$ denotes the projection map, then we have a canonical isomorphism $T_{D_y(Y)/X} \simeq g^* T_{Y/X}(-x)$.

If $Y = X$, then $D_y(Y) \simeq Y$ for any point $y \in Y(k)$. By functoriality, we see that if $s : X \to Y$ is a section of the projection map $f : Y \to X$ which passes through the point Y, then s lifts (uniquely!) to a section $\overline{s} : X \to D_y(Y)$ of the projection map $D_y(Y) \to X$.

Suppose that $f_0 : Y_0 \to X$ is a map equipped with a section s_0, and that we are given a finite sequence of points $x_1, \ldots, x_m \in X(k)$ (which need not be distinct). We can then define sequence of X-schemes $f_i : Y_i \to X$ and section $s_i : X \to Y_i$ by the formula $Y_i = D_{s_{i-1}(x_i)}(Y_{i-1})$, with s_i the unique lift of s_{i-1}. The scheme Y_m depends only on the divisor $D = x_1 + \cdots + x_m$ and the section s_0. In this case, we will say that Y_m *is obtained from* Y *by dilitation along* $s_0(D)$.

Warning 5. This is an abuse of terminology: the scheme Y_m depends not only on the set $s_0(D)$, but also on the section s_0 and the divisor D.

Remark 6. In the situation above, suppose we are given another section $s : X \to Y_0$ of the map f_0. If the sections s and s' agree on the divisor D, then s can be lifted to a map $\overline{s} : X \to Y_m$.

We will need the following algebro-geometric fact:

Proposition 7. *Let $f : Y \to X$ be a map of integral k-schemes equipped with a section h, let Z be a quasi-projective k-scheme, let $U \subseteq X$ be a dense open set, and suppose that we are given a map $\beta : U \times_X Y \to Z$ such that $\beta \circ h|_U$ can be extended to a map $X \to Z$.*

Then there exists an effective divisor $D \subseteq X$ supported in $X - U$ such that β factors as a composition

$$U \times_X Y \hookrightarrow M \xrightarrow{\overline{\beta}} Z,$$

where M denotes the scheme obtained from Y by dilitation along $h(D)$.

Let us now apply Proposition 7 to the case where $Y = G'/B' \times_{\operatorname{Spec} k} X$, $Z = \mathcal{P}/B$, β is our isomorphism $G'/B' \times_{\operatorname{Spec} k} U \simeq \mathcal{P}/B \times_X U$, and h is the zero section of the projection map $G'/B' \times_{\operatorname{Spec} k} X \to X$. It follows that there exists an effective divisor $D \subseteq X$ supported in $X - U$ and a commutative diagram

$$G'/B' \times_{\operatorname{Spec} k} X \xleftarrow{\ \phi\ } M \xrightarrow{\ \overline{\beta}\ } \mathcal{P}/B,$$

where M is obtained from $(G'/B') \times_{\operatorname{Spec} k} X$ by dilitation along $h(D)$.

Let h' be any section of the projection map $(G'/B') \times_{\operatorname{Spec} k} X$ which agrees with h on the divisor D. Then h' lifts (uniquely) to a map $\overline{h}' : X \to M$, so $\overline{\beta} \circ \overline{h}' : X \to \mathcal{P}/B$ determines a B-reduction of $\mathcal{P}$. Moreover, we have a map of vector bundles

$$\overline{h}'^* T_{M/X} \to (\overline{\beta}\overline{h}')^* T_\pi$$

on X which is an isomorphism over the open set U, and therefore induces a surjection

$$\mathrm{H}^1(X; \overline{h}'^* T_{M/X}) \to \mathrm{H}^1(X; (\overline{\beta}\overline{h}')^* T_\pi).$$

Let $\psi : (G'/B') \times_{\operatorname{Spec} k} X \to X$ denote the projection map, and let T_ψ denote the relative tangent bundle of ψ. Applying Remark 4 repeatedly, we obtain an isomorphism $T_{M/X} \simeq T_\psi(-D)$, so that

$$\mathrm{H}^1(X; \overline{h}'^* T_{M/X}) \simeq \mathrm{H}^1(X; (h'^* T_\psi)(-D)).$$

Note that h' can be identified with a map $g : X \to G'/B'$, and $h'^* T_\psi$ with the pullback $g^* T_{G'/B'}$, where $T_{G'/B'}$ denotes the tangent bundle to the flag variety G'/B'. To complete the proof of Theorem 1, it will suffice to prove the following:

Theorem 8. *Let D be an arbitrary effective divisor in X. Then there exists a map $g : X \to G'/B'$ such that $g|_D$ is constant and the cohomology group* $\mathrm{H}^1(X; g^* T_{(G'/B')}(-D))$ *vanishes.*

Fix a maximal torus $T' \subseteq B'$. Let $\Lambda^* = \mathrm{Hom}(T', \mathbf{G}_m)$ denote the character lattice of T_0, and let $\Lambda_* = \mathrm{Hom}(\mathbf{G}_m, T')$ be its cocharacter lattice. Every element $\lambda \in \Lambda^*$ determines a group homomorphism $B' \to \mathbf{G}_m$, which determines an equivariant line bundle $\mathcal{L}_\lambda$ on the flag variety G'/B'. If $g : X \to G'/B'$ is an arbitrary map, then the function $\lambda \mapsto \deg(g^* \mathcal{L}_\Lambda)$ can be regarded as an additive map from $X^*(T_0)$ to $\mathbf{Z}$, which we can identify with an element of $X_*(T_0)$. We will refer to element as the *degree* of g and denote it by $\deg(g)$.

Let Φ_- denote the set of *negative* roots of G' with respect to (B', T'): that is, the subset of Λ^* consisting of characters which appear as roots of G' but not of the Borel subgroup B'. Unwinding the definitions, we see that the tangent bundle $T_{G'/B'}$ admits a filtration whose successive quotients are the line bundles $\{\mathcal{L}_\lambda\}_{\lambda \in \Phi_-}$. Consequently, to prove the vanishing of $\mathrm{H}^1(X; g^* T_{(G'/B')}(-D))$, it will suffice to prove the vanishing of $\mathrm{H}^1(X; (g^* \mathcal{L}_\lambda)(-D))$ for $\lambda \in \Phi_-$. By the Riemann-Roch theorem, this vanishing is automatic provided that $\deg(\mathcal{L}_\lambda) > 2g - 2 + d$, where g is the genus of the curve X and d is the degree of the divisor D. We are therefore reduced to proving the following:

Theorem 9. *Let D be an arbitrary effective divisor in X and let n be a positive integer. Then there exists a map $g : X \to G'/B'$ such that $g|_D$ vanishes, and $\langle \deg(g), \lambda \rangle \geq n$ for $\lambda \in \Phi_-$.*

Choose any map $X \to \mathbf{P}^1$ which has degree $\geq n$ and is constant on the divisor D. Then any composite map $X \to \mathbf{P}^1 \stackrel{g_0}{\to} G'/B'$ is constant on D and can therefore (after translating by a point of G') be assumed to vanish on D. We are therefore reduced to proving:

Theorem 10. *There exists a map $g : \mathbf{P}^1 \to G'/B'$ such that* $\deg(g)$ *is strictly antidominant: that is,* $\langle \deg(g), \lambda \rangle > 0$ *for $\lambda \in \Phi_-$.*

Example 11. When $G' = \mathrm{SL}_2$, Theorem 10 asserts that there exists a map from $\mathbf{P}^1$ to itself of positive degree.

Example 12. Let V be the standard representation of SL_2. Then $\mathrm{Sym}^{n-1}(V)$ is an n-dimensional representation of SL_2, given by a map $\mathrm{SL}_2 \to \mathrm{SL}_n$. This map carries a Borel subgroup of SL_2 into a Borel subgroup of $G' = \mathrm{SL}_n$, and therefore induces a map of flag varieties $g : \mathbf{P}^1 \to G'/B'$. An easy calculation shows that $\langle \deg(g), \lambda \rangle = 2$ for each negative simple root λ of G', so that g satisfies the requirements of Theorem 10.

If the field k has characteristic zero, then the argument of Example 12 can be generalized by consider a "principal SL_2" in the group G'. For a general argument which works in positive characteristic, we refer the reader to [1].

References

[1] Drinfeld, V. and C. Simpson. *B-Structures on G-bundles and Local Triviality.*