Sheaves on the Ran Space (Lecture 19)

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Let k be an algebraically closed field and let ℓ be a prime number which is invertible in k. In the previous lecture, we defined for every quasi-projective k-scheme Y an ∞-category $\mathrm{Shv}_\ell(Y)$ of ℓ-adic sheaves on Y. Our goal in this lecture is to study on some infinite-dimensional algebro-geometric objects, like the Ran space of an algebraic curve X over k.

Definition 1. For every nonempty finite set T, let us consider the ∞-category $\mathrm{Shv}_\ell(X^T)$. Every surjection of finite sets $T \to T'$, let $\delta_{T/T'} : X^{T'} \to X^T$ denote the associated diagonal map. We can regard the construction

$$T \mapsto \mathrm{Shv}_\ell(X^T)$$

as a covariant functor of T in two different ways:

- (a) To each surjection of nonempty finite sets $T \to T'$, we can assign the usual pullback functor $\delta^*_{T/T'} : \mathrm{Shv}_\ell(X^T) \to \mathrm{Shv}_\ell(X^{T'})$ (that is, the left adjoint to the pushforward functor $\delta_{T/T'*}$).
- (b) To each surjection of nonempty finite sets $T \to T'$, we can assign the exceptional inverse image functor $\delta^!_{T/T'} : \mathrm{Shv}_\ell(X^T) \to \mathrm{Shv}_\ell(X^{T'})$ (that is, the right adjoint to the pushforward functor $\delta_{T/T'*}$).

We let $\mathrm{Shv}^*_\ell(\mathrm{Ran}(X))$ and $\mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$ denote the inverse limits $\varprojlim_{T \in \mathrm{Fin}^s} \mathrm{Shv}_\ell(X^T)$, where the transition maps are given by (a) and (b) respectively. We will refer to the ∞-categories $\mathrm{Shv}^*_\ell(\mathrm{Ran}(X))$ and $\mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$ as the ∞-categories of *$*$-sheaves* and *!-sheaves* on $\mathrm{Ran}(X)$, respectively.

For the remainder of this course, we will focus on the ∞-category $\mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$. By definition, an object $\mathcal{F} \in \mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$ can be described as a family of objects $\{\mathcal{F}^{(T)} \in \mathrm{Shv}_\ell(X^T)\}_{T \in \mathrm{Fin}^s}$ together with equivalences $\alpha_{T/T'} : \mathcal{F}^{(T')} \simeq \delta^!_{T/T'} \mathcal{F}^{(T)}$ associated to surjections $T \to T'$, which are compatible with composition up to coherent homotopy. Each $\alpha_{T/T'}$ can be identified with a map $\beta_{T/T'} : \delta_{T/T'*} \mathcal{F}^{(T')} \to \mathcal{F}^{(T)}$, which need not be equivalences. This motivates the following:

Definition 2. A *lax !-sheaf* on $\mathrm{Ran}(X)$ is a collection of objects $\{\mathcal{F}^{(T)} \in \mathrm{Shv}_\ell(X^T)\}_{T \in \mathrm{Fin}^s}$ together with maps $\beta_{T/T'} : \delta_{T/T'*} \mathcal{F}^{(T')} \to \mathcal{F}^{(T)}$ associated to surjections $T \to T'$, which are compatible with composition up to coherent homotopy. The collection of lax !-sheaves on $\mathrm{Ran}(X)$ can be organized into an ∞-category, which we will denote by $\mathrm{Shv}^{\mathrm{lax}}_\ell(\mathrm{Ran}(X))$. This ∞-category contains $\mathrm{Shv}^!_\ell(\mathrm{Ran}(X))$ as a full subcategory.

Example 3. For every projective k-scheme Y, let $\pi_Y : Y \to \mathrm{Spec}\, k$ denote the projection map. We let ω_Y denote $\pi^!_Y \mathbf{Z}_\ell$ (where we identify $\mathbf{Z}_\ell$ with the constant presheaf on $\mathrm{Spec}\, k$ taking the value $\mathbf{Z}_\ell$). We will refer to ω_Y as the *dualizing complex* of Y. By construction, we have a canonical equivalence

$$C_*(Y; \mathbf{Z}_\ell) \simeq C^*(Y; \mathbf{Z}_\ell)^\vee = \mathrm{Map}(\pi_{Y*} \underline{\mathbf{Z}_\ell}_Y, \mathbf{Z}_\ell) \simeq \mathrm{Map}(\underline{\mathbf{Z}_\ell}_Y, \omega_Y) = \omega_Y(Y).$$

For every proper map $f : Y \to Z$, we have a canonical equivalence $\omega_Y \simeq f^! \omega_Z$ (by functoriality). It follows that the family of sheaves $\{\omega_{X^T}\}_{T \in \mathrm{Fin}^s}$ can be regarded as a !-sheaf on $\mathrm{Ran}(X)$, which we will denote by $\omega_{\mathrm{Ran}(X)}$.

Construction 4. Let $f : Y \to Z$ be a map of quasi-projective k-schemes. Given a pair of objects $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$ and $\mathcal{G} \in \mathrm{Shv}_\ell(Z)$, any map $u : f_* \mathcal{F} \to \mathcal{G}$ induces a map on global sections

$$\mathcal{F}(Y) = (f_* \mathcal{F})(Z) \to \mathcal{G}(Z).$$

Now suppose that $\mathcal{F} = \{\mathcal{F}^{(T)}\}_{T \in \mathrm{Fin}^s}$ is a lax !-sheaf on the Ran space. Then we can regard $T \mapsto \mathcal{F}^{(T)}(X^T)$ as a contravariant functor of T. We define $\int_{\mathrm{Ran}(X)} \mathcal{F}$ to be the direct limit $\varinjlim_{T \in \mathrm{Fin}^s} \mathcal{F}^{(T)}(X^T) \in \mathrm{Mod}_{\mathbf{Z}_\ell}$. We will refer the (co)homology groups of this chain complex as the *compactly supported cohomology of* $\mathcal{F}$, or as the *chiral homology of* $\mathcal{F}$.

Example 5. The chiral homology of $\omega_{\mathrm{Ran}(X)}$ is given by

$$\int_{\mathrm{Ran}(X)} \omega_{\mathrm{Ran}(X)} = \varinjlim_{T} \omega_{X^T}(X^T) = \varinjlim_{T} C_*(X^T; \mathbf{Z}_\ell).$$

The acyclicity of the Ran space supplies a canonical equivalence $\int_{\mathrm{Ran}(X)} \omega_{\mathrm{Ran}(X)} \simeq \mathbf{Z}_\ell$.

Our next goal is to produce some examples of !-sheaves on the Ran space $\mathrm{Ran}(X)$.

Notation 6. Let Y be a proper k-scheme, and let $f : U \to Y$ be an arbitrary map. We let $[U]_Y \in \mathrm{Shv}_\ell(Y)$ denote the direct image $f_* f^* \omega_Y$.

Example 7. If $Y = \operatorname{Spec} k$, then $\omega_Y \simeq \mathbf{Z}_\ell$, and for each map $f : U \to Y$ we have $[U]_Y = f_* \underline{\mathbf{Z}_\ell}_U \simeq C^*(U; \mathbf{Z}_\ell)$ (under the equivalence $\mathrm{Shv}_\ell(Y) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}$).

In the general case, the construction $U \mapsto [U/Y]$ is contravariantly functorial in U. We can therefore extend it to prestacks as follows:

Construction 8. Let $\pi : \mathcal{C} \to \mathrm{Ring}_k$ be a prestack equipped with a map $f : \mathcal{C} \to Y$. For each object $C \in \mathcal{C}$, $\pi(C)$ is a commutative k-algebra equipped with a map of k-schemes $f_C : \operatorname{Spec} \pi(C) \to Y$, so that the sheaf $[\operatorname{Spec} \pi(C)]_Y \in \mathrm{Shv}_\ell(Y)$ is well-defined and is covariant in C. We let $[\mathcal{C}]_Y$ denote the limit $\varprojlim_{C \in \mathcal{C}} [\operatorname{Spec} \pi(C)]_Y$.

Example 9. Let $Y = \operatorname{Spec} k$. For every prestack $\mathcal{C}$, we can identify $[\mathcal{C}]_Y$ with the cochain complex $C^*(\mathcal{C}; \mathbf{Z}_\ell)$.

For fixed Y, the sheaf $[\mathcal{C}]_Y$ depends contravariantly on $\mathcal{C}$. Let us now discuss its behavior as a functor of Y. Suppose we are given a proper morphism $g : Y' \to Y$. For every map $f : U \to Y$, we can form a pullback diagram

$$\begin{array}{ccc} U' & \xrightarrow{g_U} & U \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

The proper base change theorem supplies a canonical equivalence

$$g_{U*} f'^* \omega_{Y'} \simeq f^* g_* \omega_{Y'}.$$

The equivalence $\omega_{Y'} \simeq g^! \omega_Y$ is adjoint to a map $g_* \omega_{Y'} \to \omega_Y$. Composing with this map and applying the functor f_*, we obtain a map

$$\begin{aligned} g_*[U']_{Y'} &= g_* f'_* f'^* \omega_{Y'} \\ &\simeq f_* g_{U*} f'^* \omega_{Y'} \\ &\simeq f_* f^* g_* \omega_{Y'} \\ &\to f_* f^* \omega_Y \\ &= [U]_Y. \end{aligned}$$

More generally, if $\mathcal{C}$ is a prestack over Y and $\mathcal{C}' = Y' \times_Y \mathcal{C}$, then (by passing to the inverse limit) the above construction supplies a canonical map $g_*[\mathcal{C}']_{Y'} \to [\mathcal{C}]_Y$, which we can identify with a map $[\mathcal{C}']_{Y'} \to g^! [\mathcal{C}]_Y$.

Remark 10. In the above situation, if the map $U \to Y$ is smooth, then the induced map $[U']_{Y'} \to g^{!}[U]_Y$ is an equivalence; this is a consequence of the smooth base change theorem for étale cohomology. In this case, for each point $y \in Y(k)$, if $i_y : \operatorname{Spec} k \hookrightarrow Y$ denotes the closed immersion determined by y, then we can identify $i_y^{!}[U]_Y$ with the cochain complex $C^*(U_y; \mathbf{Z}_\ell)$. In other words, $[U]_Y$ is a sheaf on Y whose costalks compute the cohomology of the map $U \to Y$.