Acyclicity of the Ran Space (Lecture 10)

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Throughout this lecture, we let k denote an algebraically closed field, ℓ a prime number which is invertible in k, and $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}\}$. Suppose we are given an algebraic curve X over k and a smooth affine group scheme G over X. In the previous lecture, we defined a map of prestacks

$$\mathrm{Ran}_G(X) \to \mathrm{Bun}_G(X),$$

and asserted that it induces an isomorphism on homology provided that the generic fiber of G is semisimple and simply connected. Our goal in this lecture is to prove this statement in the special case where G is trivial. In this case, $\mathrm{Bun}_G(X) \simeq \mathrm{Spec}\, k$, so we asserting that the prestack $\mathrm{Ran}_G(X) = \mathrm{Ran}(X)$ is *acyclic*: that is, it has the homology of a point. To formulate and prove this assertion, we can get away with much weaker assumptions on X.

Theorem 1 (Beilinson-Drinfeld)**.** *Let X be a quasi-projective k-scheme. If X is connected, then the canonical map*

$$\mathrm{Ran}(X) \to \mathrm{Spec}\, k$$

induces an isomorphism on homology with coefficients in Λ. In other words, we have canonical isomorphisms

$$\mathrm{H}_*(\mathrm{Ran}(X); \Lambda) \simeq \begin{cases} \Lambda & \text{if } * = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Notation 2. For any prestack $\mathcal{C}$, we let $C_*^{\mathrm{red}}(\mathcal{C}; \Lambda)$ denote the fiber of the canonical map $C_*(\mathcal{C}; \Lambda) \to C_*(\mathrm{Spec}\, k; \Lambda)$. We will refer to $C_*^{\mathrm{red}}(\mathcal{C}; \Lambda)$ as the *reduced chain complex of* $\mathcal{C}$ with coefficients in Λ. Theorem 1 is equivalent to the assertion that the chain complex $C_*^{\mathrm{red}}(\mathrm{Ran}(X); \Lambda)$ is acyclic.

Remark 3. For any prestack $\mathcal{C}$, we have a canonical equivalence

$$C_*^{\mathrm{red}}(\mathcal{C}; \Lambda) \simeq \Lambda \otimes_{\mathbf{Z}_\ell} C_*^{\mathrm{red}}(\mathcal{C}; \mathbf{Z}_\ell).$$

Consequently, to prove Theorem 1, it will suffice to treat the case where $\Lambda = \mathbf{Z}_\ell$.

Note that we have a canonical fiber sequence

$$C_*^{\mathrm{red}}(\mathcal{C}; \mathbf{Z}_\ell) \xrightarrow{\ell} C_*^{\mathrm{red}}(\mathcal{C}; \mathbf{Z}_\ell) \to C_*^{\mathrm{red}}(\mathcal{C}; \mathbf{Z}/\ell\mathbf{Z}).$$

If $C_*^{\mathrm{red}}(\mathcal{C}; \mathbf{Z}/\ell\mathbf{Z})$ is acyclic, then multiplication by ℓ induces a quasi-isomorphism from the chain complex $C_*^{\mathrm{red}}(\mathcal{C}; \mathbf{Z}_\ell)$ to itself. In this case, we have $C_*^{\mathrm{red}}(\mathcal{C}; \mathbf{Z}_\ell) \simeq C_*^{\mathrm{red}}(\mathcal{C}; \mathbf{Q}_\ell)$. It will therefore suffice to prove Theorem 1 in the special case where $\Lambda \in \{\mathbf{Q}_\ell, \mathbf{Z}/\ell\mathbf{Z}\}$ is a field.

We begin by establishing the following special case of Theorem 1:

Proposition 4. *Suppose that X is connected. Then the map $\mathrm{Ran}^u(X) \to \mathrm{Spec}\, k$ induces an isomorphism of abelian groups $\mathrm{H}_0(\mathrm{Ran}^u(X); \Lambda) \to \mathrm{H}_0(\mathrm{Spec}\, k; \Lambda) \simeq \Lambda$.*

Proof of Proposition 4. Let $\mathrm{Fin}^{\mathrm{s}}$ denote the category whose objects are nonempty finite sets and whose morphisms are surjections. The construction $(R, S, \mu) \mapsto S$ determines a fibration of categories $\mathrm{Ran}(X) \to \mathrm{Fin}^{\mathrm{s}}$ (that is, a functor whose opposite is an op-fibration), whose fiber over an object $S \in \mathrm{Fin}^{\mathrm{s}}$ can be identified with X^S (as a prestack). It follows from general nonsense that we have an equivalence

$$C_*(\mathrm{Ran}(X); \Lambda) \simeq \varinjlim_{S \in \mathrm{Fin}^{\mathrm{s}}} C_*(X^S; \Lambda)$$

in the ∞-category Mod_Λ. Using the fact that these chain complexes have homologies only in nonnegative degrees, we obtain an isomorphism

$$\mathrm{H}_0(\mathrm{Ran}(X); \Lambda) \simeq \varinjlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathrm{H}_0(X^S; \Lambda)$$

in the ordinary category of abelian groups. Since X is connected, each of the groups $\mathrm{H}_0(X^S; \Lambda)$ is isomorphic to Λ. Since the category $\mathrm{Fin}^{\mathrm{s}}$ is connected (for example, it has a final object), it follows that the colimit $\varinjlim_{S \in \mathrm{Fin}^{\mathrm{s}}} \mathrm{H}_0(X^S; \Lambda)$ is isomorphic to Λ. □

To prove Theorem 1 in general, it will be convenient to work with a slightly different incarnation of the Ran space $\mathrm{Ran}(X)$:

Definition 5. Let X be a quasi-projective k-scheme. We define a category $\mathrm{Ran}^u(X)$ as follows:

(1) The objects of $\mathrm{Ran}^u(X)$ are pairs (R, S) where R is a finitely generated k-algebra and S is a nonempty finite subset of $X(R)$.

(2) A morphism from (R, S) to (R', S') in $\mathrm{Ran}^u(X)$ is a k-algebra homomorphism $\phi : R \to R'$ having the property that S' is the image of the induced map $S \subseteq X(R) \xrightarrow{X(\phi)} X(R')$.

The construction $(R, S) \mapsto R$ determines a forgetful functor $\mathrm{Ran}^u(X) \to \mathrm{Ring}_k$. It is easy to see that this functor is an op-fibration, so that we can regard $\mathrm{Ran}^u(X)$ as a prestack. We will refer to $\mathrm{Ran}^u(X)$ as the *unlabelled Ran space of* X.

Note that the fibers of the forgetful functor $\mathrm{Ran}^u(X) \to \mathrm{Ring}_k$ are discrete (that is, they have only identity morphisms). We may therefore identify $\mathrm{Ran}^u(X)$ with a $\mathcal{S}\mathrm{et}$-valued functor on the category Ring_k: namely, the functor

$$R \mapsto \{\emptyset \neq S \subseteq X(R) | \ S \text{ is finite } \}.$$

More informally: $\mathrm{Ran}^u(X)$ is the prestack of nonempty finite subsets of X.

Remark 6. The constructions $\mathcal{C} \mapsto C_*(\mathcal{C}; \Lambda)$ and $\mathcal{C} \mapsto C^*(\mathcal{C}; \Lambda)$ determine functors from from the 2-category of categories with a map $\pi : \mathcal{C} \to \mathrm{Ring}_k$ to the ∞-category Mod_Λ. In particular, if $F, G : \mathcal{C} \to \mathcal{C}'$ maps of prestacks and α is a morphism from F to G, then α induces a chain homotopy between the induced maps $C_*(F; \Lambda), C_*(G; \Lambda) : C_*(\mathcal{C}; \Lambda) \to C_*(\mathcal{C}'; \Lambda)$, so that F and G induce the *same* map from $\mathrm{H}_*(\mathcal{C}; \Lambda)$ to $\mathrm{H}_*(\mathcal{C}'; \Lambda)$ (and similarly for cohomology). Note that this does *not* require that α is invertible, or that F and G preserve coCartesian morphisms.

Example 7. The construction $(R, S, \mu) \mapsto (R, \mu(S))$ determines a map of prestacks $F : \mathrm{Ran}(X) \to \mathrm{Ran}^u(X)$. This map has a right inverse, given by the functor $G : \mathrm{Ran}^u(X) \to \mathrm{Ran}(X)$ carrying (R, S) to (R, S, id) (the morphism G is not a map of prestacks: that is, it does not preserve coCartesian morphisms). The composition $G \circ F : \mathrm{Ran}(X) \to \mathrm{Ran}(X)$ is not isomorphic to the identity. However, there is a natural transformation $\mathrm{id} \to G \circ F$, which carries an object $(R, S, \mu) \in \mathrm{Ran}(X)$ to the commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & \mu(S) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mathrm{id}} \\ X(R) & \xrightarrow{\mathrm{id}} & X(R). \end{array}$$

It follows from Remark 6 that F and G induce mutually inverse isomorphisms between $\mathrm{H}_*(\mathrm{Ran}(X);\Lambda)$ and $\mathrm{H}_*(\mathrm{Ran}^u(X);\Lambda)$.

We will need the following fact:

Proposition 8 (Künneth Formula for Prestacks)**.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be prestacks. Then we have a canonical equivalence*

$$C_*(\mathcal{C};\Lambda)\otimes_\Lambda C_*(\mathcal{C}';\Lambda)\simeq C_*(\mathcal{C}\times_{\mathrm{Ring}_k}\mathcal{C}';\Lambda)$$

in the ∞-category Mod_Λ.

Remark 9. In the special case where $\Lambda\in\{\mathbf{Q}_\ell,\mathbf{Z}/\ell\mathbf{Z}\}$ is a field, the formation of tensor products over Λ is compatible with passage to homology. We therefore obtain a Künneth formula in homology:

$$\mathrm{H}_*(\mathcal{C};\Lambda)\otimes_\Lambda \mathrm{H}_*(\mathcal{C}';\Lambda)\simeq \mathrm{H}_*(\mathcal{C}\times_{\mathrm{Ring}_k}\mathcal{C}';\Lambda)$$

Proof of Theorem 1. Using Remark 3, we may assume that Λ is a field. By virtue of Proposition 4, it will suffice to show that $\mathrm{H}^{\mathrm{red}}_n(\mathrm{Ran}^u(X);\Lambda)\simeq 0$ for $n\geq 0$. We proceed by induction on n; in the case $n=0$, this follows from Proposition 4. Let us therefore assume that $n>0$ and that $\mathrm{H}^{\mathrm{red}}_i(\mathrm{Ran}^u(X);\Lambda)\simeq 0$ for $i<n$. Set $V=\mathrm{H}_n(\mathrm{Ran}^u(X);\Lambda)$. Using Remark 9, we obtain an isomorphism

$$\mathrm{H}_n(\mathrm{Ran}^u(X)\times_{\mathrm{Spec}\,k}\mathrm{Ran}^u(X);\Lambda)\simeq V\oplus V.$$

We have an evident map $m:\mathrm{Ran}^u(X)\times_{\mathrm{Spec}\,k}\mathrm{Ran}^u(X)\to\mathrm{Ran}^u(X)$, given on objects by the formula

$$((R,S),(R,S'))\mapsto(R,S\cup S').$$

Passing to homology, we obtain a map $V\oplus V\to V$, which we can identify with a pair of maps $\lambda,\mu:V\to V$. By symmetry, we have $\lambda=\mu$. Note that the composite map

$$\mathrm{Ran}^u(X)\xrightarrow{\delta}\mathrm{Ran}^u(X)\times_{\mathrm{Spec}\,k}\mathrm{Ran}^u(X)\xrightarrow{m}\mathrm{Ran}^u(X)$$

is the identity. From this, we deduce that $v=\lambda(v)+\mu(v)=2\lambda(v)$ for $v\in V$.

Choose a k-rational point $x\in X$. Then $\{x\}$ can be identified with a k-rational point of $\mathrm{Ran}^u(X)$: that is, with a map of prestacks $\iota:\mathrm{Spec}\,k\to\mathrm{Ran}^u(X)$. Let F denote the composite map

$$\mathrm{Ran}^u(X)\simeq\mathrm{Spec}\,k\times_{\mathrm{Spec}\,k}\mathrm{Ran}^u(X)\xrightarrow{(\iota,\mathrm{id})}\mathrm{Ran}^u(X)\times_{\mathrm{Spec}\,k}\mathrm{Ran}^u(X)\xrightarrow{m}\mathrm{Ran}^u(X).$$

Passing to homology, we see that F induces a map from V to V given by $v\mapsto\lambda(v)$. Since $F^2=F$, we have

$$2\lambda(v)=2\lambda(\lambda(v))=\lambda(v),$$

so that $\lambda(v)=0$ and therefore $v=2\lambda(v)=0$. Since this is true for all $v\in V$, we conclude that $V\simeq 0$. $\square$

Proof of Proposition 8. Suppose that we are given a pair of prestacks $\pi:\mathcal{C}\to\mathrm{Ring}_k$ and $\pi':\mathcal{C}'\to\mathrm{Ring}_k$. Then $\mathcal{C}$ and $\mathcal{C}'$ have a product in the 2-category of prestacks, given by the fiber product $\mathcal{C}\times_{\mathrm{Ring}_k}\mathcal{C}'$. We can also consider the usual Cartesian product of $\mathcal{C}$ and $\mathcal{C}'$, which is a category $\mathcal{C}\times\mathcal{C}'$.

Given a pair of objects $C\in\mathcal{C}$ and $C'\in\mathcal{C}'$, the we can lift the inclusion map $\pi(C)\to\pi(C)\otimes_k\pi(C')$ to a π-coCartesian map $C\to D$ in $\mathcal{C}$, and the other inclusion $\pi(C')\to\pi(C)\otimes_k\pi(C')$ to a π'-coCartesian map $C'\to D'$ in $\mathcal{C}'$. Then D and D' have the same image $\pi(C)\otimes_k\pi'(C')$, so that (D,D') can be regarded as an object of $\mathcal{C}\times_{\mathrm{Ring}_k}\mathcal{C}'$. The construction $(C,C')\mapsto(D,D')$ determines a functor $G:\mathcal{C}\times\mathcal{C}'\to\mathcal{C}\times_{\mathrm{Ring}_k}\mathcal{C}'$, which induces a map

$$\varinjlim_{C\in\mathcal{C},C'\in\mathcal{C}'} C_*(\mathrm{Spec}\,\pi(C)\otimes_k\pi'(C');\Lambda)\to C_*(\mathcal{C}\times_{\mathrm{Ring}_k}\mathcal{C}';\Lambda).$$

In fact, this map is an equivalence: it has a homotopy inverse given by

$$\begin{aligned} C_*(\mathcal{C} \times_{\mathrm{Ring}_k} \mathcal{C}'; \Lambda) &= \varinjlim_{\pi(C)=\pi'(C')} C_*(\operatorname{Spec} \pi(C); \Lambda) \\ &\to \varinjlim_{\pi(C)=\pi'(C')} C_*(\operatorname{Spec} \pi(C) \otimes_k \pi'(C'); \Lambda) \\ &\to \varinjlim_{C \in \mathcal{C}, C' \in \mathcal{C}'} C_*(\operatorname{Spec} \pi(C) \otimes_k \pi'(C'); \Lambda). \end{aligned}$$

Using the Künneth formula for affine k-schemes, we can compute

$$\begin{aligned} C_*(\mathcal{C} \times_{\mathrm{Ring}_k} \mathcal{C}'; \Lambda) &\simeq \varinjlim_{C \in \mathcal{C}, C' \in \mathcal{C}'} C_*(\operatorname{Spec} \pi(C) \otimes_k \pi'(C'); \Lambda) \\ &\simeq \varinjlim_{C \in \mathcal{C}, C' \in \mathcal{C}'} C_*(\operatorname{Spec} \pi(C); \Lambda) \otimes_\Lambda C_*(\operatorname{Spec} \pi'(C'); \Lambda) \\ &\simeq (\varinjlim_{C \in \mathcal{C}} C_*(\operatorname{Spec} \pi(C); \Lambda)) \otimes_\Lambda (\varinjlim_{C' \in \mathcal{C}'} C_*(\operatorname{Spec} \pi'(C'); \Lambda)) \\ &= C_*(\mathcal{C}; \Lambda) \otimes_\Lambda C_*(\mathcal{C}'; \Lambda). \end{aligned}$$

□

Remark 10. The key point in our deduction of Proposition 8 was that we could replace a colimit over the category $\mathcal{C} \times_{\mathrm{Ring}_k} \mathcal{C}'$ by a colimit over the Cartesian product $\mathcal{C} \times \mathcal{C}'$. This is because the functor $G : \mathcal{C} \times \mathcal{C}' \to \mathcal{C} \times_{\mathrm{Ring}_k} \mathcal{C}'$ is right adjoint to the inclusion map $\mathcal{C} \times_{\mathrm{Ring}_k} \mathcal{C}' \to \mathcal{C} \times \mathcal{C}'$, and therefore a *cofinal* functor. For more details, see [2].

Warning 11. The analogous Künneth formula does not necessarily hold for cochain complexes $C^*(\mathcal{C}; \Lambda)$, because in general the formation of tensor products does not distribute over inverse limits. This is one reason that it will be convenient for us to work with the homology of prestacks.

References

[1] Beilinson, A. and V. Drinfeld. *Chiral algebras.* American Mathematical Society Colloquium Publications 51. American Mathematical Society, Providence, RI, 2004.

[2] Lurie, J. *Higher Topos Theory.* Princeton University Press, 2009.