Homology and Cohomology of Stacks (Lecture 7)

February 19, 2014

In this course, we will need to discuss the ℓ-adic homology and cohomology of algebro-geometric objects of a more general nature than algebraic varieties: for example, the moduli stack $\mathrm{Bun}_G(X)$ of G-bundles on an algebraic curve. Let us therefore briefly review the language of stacks.

Let Ring_k denote the category of finitely generated k-algebras. If X is a k-scheme and $R \in \mathrm{Ring}_k$, then an *R-valued point* of X is a map $\mathrm{Spec}\, R \to X$ in the category of k-schemes. The collection of all R-valued points of X forms a set $X(R)$. The construction $R \mapsto X(R)$ determines a functor from Ring_k to the category of sets. We refer to this functor as the *functor of points* of X. If X is of finite type over k (or if we were to enlarge Ring_k to include k-algebras which are not finitely generated), then X is determined by its functor of points up to canonical isomorphism. In this case, we will generally abuse notation by identifying X with its functor of points.

Now suppose that G is a smooth affine group scheme over an algebraic curve X. We would like to introduce an algebro-geometric object $\mathrm{Bun}_G(X)$ which classifies G-bundles on X. In other words, we would like the R-points of $\mathrm{Bun}_G(X)$ to be G-bundles on the relative curve $X_R = \mathrm{Spec}\, R \times_{\mathrm{Spec}\, k} X$. Here some caution is in order. The collection of all G-bundles on X_R naturally forms a category, rather than a set. Let us denote this category by $\mathrm{Bun}_G(X)(R)$. If $\phi : R \to R'$ is a k-algebra homomorphism, then ϕ determines a map of categories $\phi^* : \mathrm{Bun}_G(X)(R) \to \mathrm{Bun}_G(X)(R')$, given on objects by the formula

$$\phi^* \mathcal{P} = X_{R'} \times_{X_R} \mathcal{P}.$$

However, this construction is not strictly functorial: given another ring homomorphism $\psi : R' \to R''$, the iterated pullback

$$\psi^*(\phi^* \mathcal{P}) = X_{R''} \times_{X_{R'}} (X_{R'} \times_{X_R} \mathcal{P})$$

is canonically isomorphic to $X_{R''} \times_{X_R} \mathcal{P}$, but it is unnatural to expect that they should be literally identical to one another.

It is possible to axiomatize the functorial behavior exhibited by the construction $R \mapsto \mathrm{Bun}_G(X)(R)$ using the language of 2-categories. However, it is often more convenient to encode the same data in a different package, where the functoriality is "implicit" rather than "explicit".

Definition 1. Let X be an algebraic curve over k and let G be a smooth group scheme over X. We define a category $\mathrm{Bun}_G(X)$ as follows:

(1) The objects of $\mathrm{Bun}_G(X)$ are pairs $(R, \mathcal{P})$, where R is a finitely presented k-algebra and $\mathcal{P}$ is a G-bundle on the relative curve $X_R = \mathrm{Spec}\, R \times_{\mathrm{Spec}\, k} X$.

(2) A morphism from $(R, \mathcal{P})$ to $(R', \mathcal{P}')$ consists of a k-algebra homomorphism $\phi : R \to R'$ together with a G-bundle isomorphism α between $\mathcal{P}'$ and $X_{R'} \times_{X_R} \mathcal{P}$.

We will refer to $\mathrm{Bun}_G(X)$ as the *moduli stack of G-bundles*.

By construction, the construction $(R, \mathcal{P}) \mapsto R$ determines a forgetful functor $\pi : \mathrm{Bun}_G(X) \to \mathrm{Ring}_k$. Moreover, for every finitely generated k-algebra R, we can recover the category $\mathrm{Bun}_G(X)(R)$ of G-bundles on X_R as the fiber product $\mathrm{Bun}_G(X) \times_{\mathrm{Ring}_k} \{R\}$. Moreover, the map π also encodes the functoriality of the construction $R \mapsto \mathrm{Bun}_G(X)(R)$: given an object $(R, \mathcal{P}) \in \mathrm{Bun}_G(X)(R)$ and a ring homomorphism $\phi : R \to R'$, we can choose any lift of ϕ to a morphism $(\phi, \alpha) : (R, \mathcal{P}) \to (R', \mathcal{P}')$ in $\mathrm{Bun}_G(X)$. Such a lift then exhibits $\mathcal{P}'$ as a fiber product $X_{R'} \times_{X_R} \mathcal{P}$.

More generally, for any functor $\pi : \mathcal{C} \to \mathcal{D}$ and any object $D \in \mathcal{D}$, let $\mathcal{C}_D$ denote the fiber product $\mathcal{C} \times_{\mathcal{D}} \{D\}$. We might then ask if $\mathcal{C}_D$ depends functorially on D, in some sense. This requires an assumption on the functor π.

Definition 2. Let $\pi : \mathcal{C} \to \mathcal{D}$ be a functor between categories. We say that a morphism $\alpha : C \to C'$ in $\mathcal{C}$ is *π-coCartesian* if, for every object $C'' \in \mathcal{C}$, composition with α induces a bijection

$$\mathrm{Hom}_{\mathcal{C}}(C', C'') \to \mathrm{Hom}_{\mathcal{C}}(C, C'') \times_{\mathrm{Hom}_{\mathcal{D}}(\pi C, \pi C'')} \mathrm{Hom}_{\mathcal{D}}(\pi C', \pi C'').$$

We will say that π is an *op-fibration* (also called a *Grothendieck op-fibration*) if, for every object $C \in \mathcal{C}$ and every morphism $\alpha_0 : \pi C \to D$ in the category $\mathcal{D}$, there exists an π-coCartesian morphism $\alpha : C \to \overline{D}$ with $\alpha_0 = \pi(\alpha)$.

Let k be an algebraically closed field, and let Ring_k denote the category of finitely generated k-algebras. We define a *prestack* to be a category $\mathcal{C}$ equipped with a Grothendieck op-fibration $\pi : \mathcal{C} \to \mathrm{Ring}_k$.

Remark 3. We will generally abuse notation by identifying a prestack $\pi : \mathcal{C} \to \mathrm{Ring}_k$ with its underlying category $\mathcal{C}$ and simply say that $\mathcal{C}$ is a prestack, or that π *exhibits $\mathcal{C}$ as a prestack*.

Example 4. The forgetful functor $\mathrm{Bun}_G(X) \to \mathrm{Ring}_k$ is an op-fibration, and therefore exhibits $\mathrm{Bun}_G(X)$ as a prestack.

Example 5. Let X be a k-scheme. We can associate to X a category $\mathcal{C}_X$, which we call the *category of points* of X. By definition, an object of $\mathcal{C}_X$ is a pair (R, ϕ), where R is a finitely presented k-algebra and $\phi : \mathrm{Spec}\, R \to X$ is a map of k-schemes. A morphism from (R, ϕ) to (R', ϕ') is a k-algebra homomorphism $\psi : R \to R'$ for which the diagram

$$\begin{array}{ccc} \mathrm{Spec}\, R' & \xrightarrow{\mathrm{Spec}(\psi)} & \mathrm{Spec}\, R \\ & {\scriptstyle \phi'} \searrow \quad \swarrow {\scriptstyle \phi} & \\ & X & \end{array}$$

commutes. The construction $(R, \phi) \mapsto R$ defines a forgetful functor $\mathcal{C}_X \to \mathrm{Ring}_k$, which exhibits $\mathcal{C}_X$ as a prestack.

Example 6 (Grothendieck Construction)**.** Let $\mathcal{D}$ be a category, and let U be a functor from $\mathcal{D}$ to the category $\mathcal{C}\mathrm{at}$ of categories. We can define a new category $\mathcal{D}_U$ as follows:

(1) The objects of $\mathcal{D}_U$ are pairs (D, u) where D is an object of $\mathcal{D}$ and u is an object of the category $U(D)$.

(2) A morphism from (D, u) to (D', u') consists of a pair (ϕ, α), where $\phi : D \to D'$ is a morphism in $\mathcal{D}$ and $\alpha : U(\phi)(u) \to u'$ is a morphism in $U(D')$.

The construction $(D, u) \mapsto D$ determines a forgetful functor $\mathcal{D}_U \to \mathcal{D}$ which is a Grothendieck op-fibration. The passage from U to $\mathcal{D}_U$ is often called the *Grothendieck construction*.

For any Grothendieck op-fibration $F : \mathcal{C} \to \mathcal{D}$, the category $\mathcal{C}$ is equivalent to $\mathcal{D}_U$, for some functor $U : \mathcal{D} \to \mathcal{C}\mathrm{at}$. Moreover, the data of F and the data of the functor U are essentially equivalent to one another (in a suitable 2-categorical sense).

Remark 7. Let $\pi : \mathcal{C} \to \mathrm{Ring}_k$ and $\pi' : \mathcal{C}' \to \mathrm{Ring}_k$ be prestacks. A *map of prestacks* from $\mathcal{C}$ to $\mathcal{C}'$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ which carries π-coCartesian morphisms to π'-coCartesian morphisms, and for which the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\ & \searrow \quad \swarrow & \\ & \mathrm{Ring}_k & \end{array}$$

commutes. The collection of all maps of prestacks from $\mathcal{C}$ to $\mathcal{C}'$ is organized into a category $\mathrm{Hom}(\mathcal{C}, \mathcal{C}')$, where a morphism from $F : \mathcal{C} \to \mathcal{C}'$ to $G : \mathcal{C} \to \mathcal{C}'$ is a natural transformation of functors $\alpha : F \to G$ such that, for each object $C \in \mathcal{C}$, the map $\pi'(\alpha_C)$ is an identity morphism in Ring_k. If $\pi'' : \mathcal{C}'' \to \mathrm{Ring}_k$ is another prestack, we have evident composition functors

$$\mathrm{Hom}(\mathcal{C}, \mathcal{C}') \times \mathrm{Hom}(\mathcal{C}', \mathcal{C}'') \to \mathrm{Hom}(\mathcal{C}, \mathcal{C}'').$$

We can summarize the situation by saying that the collection of all k-prestacks forms a (strict) 2-category.

Remark 8. We can consider a more general notion of k-prestack morphism, given by a diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}' \\ & \searrow \quad \swarrow & \\ & \mathrm{Ring}_k & \end{array}$$

which commutes only up to *specified* isomorphism. The collection of all such morphisms can be organized into a category $\mathrm{Hom}'(\mathcal{C}, \mathcal{C}')$, which contains $\mathrm{Hom}(\mathcal{C}, \mathcal{C}')$ as a subcategory. However, it is not hard to see that the inclusion $\mathrm{Hom}(\mathcal{C}, \mathcal{C}') \hookrightarrow \mathrm{Hom}'(\mathcal{C}, \mathcal{C}')$ is an equivalence of categories, so there is no real gain in generality.

Remark 9. Let X and Y be k-schemes, and let $\mathrm{Hom}_k(X, Y)$ be the *set* of k-scheme maps from X to Y. We regard $\mathrm{Hom}_k(X, Y)$ as a category, having no morphisms other than the identities. We have an evident functor $\mathrm{Hom}_k(X, Y) \to \mathrm{Hom}(\mathcal{C}_X, \mathcal{C}_Y)$. If X is locally of finite type over k, then this map is an isomorphism of categories. In particular, the construction $X \mapsto \mathcal{C}_X$ determines a fully faithful embedding from the 1-category of schemes which are locally of finite type over k to the 2-category of prestacks. In other words, if X is a scheme which is locally of finite type over k, then X can be functorially reconstructed from the associated prestack $\mathcal{C}_X$. Because of this, we will generally abuse notation by identifying X with the prestack $\mathcal{C}_X$.

Example 10. Let R be a finitely generated k-algebra, and let $\operatorname{Spec} R$ be the associated affine scheme. Then the prestack associated to $\operatorname{Spec} R$ is the category Ring_R of finitely generated R-algebras (viewed as a prestack via the functor $\mathrm{Ring}_R \to \mathrm{Ring}_k$ which "forgets" the R-algebra structure).

Definition 11. Let $\pi : \mathcal{C} \to \mathrm{Ring}_k$ be a prestack. We say that $\mathcal{C}$ is a *prestack in groupoids* if every morphism of $\mathcal{C}$ is π-coCartesian.

Remark 12. The condition that a prestack $\pi : \mathcal{C} \to \mathrm{Ring}_k$ be a prestack in groupoids is equivalent to the condition that for each $R \in \mathrm{Ring}_k$, the category $\mathcal{C}_R = \mathcal{C} \times_{\mathrm{Ring}_k} \{R\}$ is a groupoid (that is, every morphism in $\mathcal{C}_R$ is an equivalence).

Example 13. All of the prestacks we have considered so far are prestacks in groupoids. For example, the prestack $\mathcal{C}_Y$ associated to a k-scheme Y, the moduli stack $\mathrm{Bun}_G(X)$, and the prestack $\mathrm{Ran}^u_G(X)$ are prestacks in groupoids. The prestack $\mathrm{Ran}(X)$ introduced in the previous lecture is *not* a prestack in groupoids.

Remark 14. In addition to the notion of prestack we have introduced, there is a more restrictive notion of *stack*. By definition, a stack (for the étale topology) is a prestack $\pi : \mathcal{C} \to \mathrm{Ring}_k$ for which the construction $R \mapsto \mathcal{C}_R$ satisfies descent (for the étale topology). Here we could replace the étale topology on Sch_k by any other Grothendieck topology (Zariski, flat, Nisnevich, etcetera). All of the examples of prestacks that we have considered so far are stacks (for any of these topologies).

Example 15. Let X be a quasi-projective k-scheme. We define a category $\mathrm{Ran}(X)$ as follows:

- The objects of $\mathrm{Ran}(X)$ are triples (R, S, μ) where R is a finitely generated k-algebra, S is a nonempty finite set, and $\mu : S \to X(R)$ is a map of sets.
- A morphism from (R, S, μ) to (R', S', μ') in $\mathrm{Ran}(X)$ consists of a k-algebra homomorphism $R \to R'$ and a surjection of finite sets $S \to S'$ for which the diagram

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} \\ X(R) & \longrightarrow & X(R') \end{array}$$

  commutes.

The construction $\mathrm{Ran}(X) \to \mathrm{Ring}_k$ is an op-fibration, so we can regard $\mathrm{Ran}(X)$ as a prestack. We will refer to $\mathrm{Ran}(X)$ as the *Ran space* of X.

More informally, $\mathrm{Ran}(X)$ is a prestack whose R-valued points are finite sets of R-valued points of X. Note that $\mathrm{Ran}(X)$ is *not* a prestack in groupoids (since a surjection of finite sets need not be bijective), and is *not* a stack for the étale topology on Sch_k (or even the Zariski topology).

Definition 16. Let ℓ be a prime number which is invertible in k, and let $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}\}$.

For any prestack $\pi : \mathcal{C} \to \mathrm{Ring}_k$, we define chain complexes $C^*(\mathcal{C}; \Lambda)$ and $C_*(\mathcal{C}; \Lambda)$ by the formulae

$$C^*(\mathcal{C}; \Lambda) = \varprojlim_{C \in \mathcal{C}} C^*(\mathrm{Spec}\, \pi(C); \Lambda) \qquad C_*(\mathcal{C}; \Lambda) = \varinjlim_{C \in \mathcal{C}} C_*(\mathrm{Spec}\, \pi(C); \Lambda).$$

Here the limit and colimit are taken in the ∞-category Mod_Λ.

We let $\mathrm{H}^*(\mathcal{C}; \Lambda)$ denote the cohomology groups of $C^*(\mathcal{C}; \Lambda)$, and $\mathrm{H}_*(\mathcal{C}; \Lambda)$ the homology groups of $C_*(\mathcal{C}; \Lambda)$. We refer to $\mathrm{H}^*(\mathcal{C}; \Lambda)$ as the *étale cohomology of* $\mathcal{C}$ *with coefficients in* Λ, and $\mathrm{H}_*(\mathcal{C}; \Lambda)$ as the *étale homology of* $\mathcal{C}$ *with coefficients in* Λ.

Example 17. Let $X \in \mathrm{Sch}_k$ be a quasi-projective k-scheme, and let $\mathcal{C}_X$ be the associated prestack. If $X = \mathrm{Spec}\, R$ is affine, then the category $\mathcal{C}_X$ has a final object (given by the pair (R, id)), so we have canonical equivalences

$$C^*(\mathcal{C}_X; \Lambda) \simeq C^*(\mathrm{Spec}\, R; \Lambda) \qquad C_*(\mathcal{C}_X; \Lambda) \simeq C_*(\mathrm{Spec}\, R; \Lambda).$$

By construction, the functor $X \mapsto C^*(X; \Lambda)$ satisfies descent for the étale topology. From this, one can deduce the existence of general equivalences

$$C^*(\mathcal{C}_X; \Lambda) \simeq C^*(X; \Lambda) \qquad C_*(\mathcal{C}_X; \Lambda) \simeq C_*(X; \Lambda).$$

Warning 18. Let $\mathcal{C}$ be a prestack. Then $C^*(\mathcal{C}; \Lambda)$ can be identified with the Λ-linear dual of $C_*(\mathcal{C}; \Lambda)$. In particular, if Λ is a field, then we have canonical isomorphisms

$$\mathrm{H}^i(\mathcal{C}; \Lambda) \simeq \mathrm{H}_i(\mathcal{C}; \Lambda)^\vee.$$

However, $C_*(\mathcal{C}; \Lambda)$ need not be the Λ-linear dual of $C^*(\mathcal{C}; \Lambda)$ (if Λ is a field, this is true if and only if each $\mathrm{H}_i(\mathcal{C}; \Lambda)$ is a finite-dimensional vector space).

Warning 19. Let $\mathcal{C}$ be a prestack. Then $C_*(\mathcal{C}; \mathbf{Q}_\ell) = C_*(\mathcal{C}; \mathbf{Z}_\ell)[\ell^{-1}]$, since the process of "inverting ℓ" commutes with colimits. However, it generally does not commute with limits, so that the canonical map

$$C^*(\mathcal{C}; \mathbf{Z}_\ell)[\ell^{-1}] \rightarrow C^*(\mathcal{C}; \mathbf{Q}_\ell)$$

is not an equivalence in general.

Warning 20. For every prestack $\mathcal{C}$, we have a canonical equivalence $C^*(\mathcal{C}; \mathbf{Z}_\ell) \simeq \varprojlim C^*(\mathcal{C}; \mathbf{Z}/\ell^d\mathbf{Z})$. However, the canonical map $C_*(\mathcal{C}; \mathbf{Z}_\ell) \rightarrow \varprojlim C_*(\mathcal{C}; \mathbf{Z}/\ell^d\mathbf{Z})$ need not be an equivalence in general.

Using Definition 16, we can precise formulate the main problem we will attempt to address in this course:

Question 21. Let X be an algebraic curve over k and let G be a smooth affine group scheme over X. What can we say about the cohomology $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$?

References

[1] Freitag, E. and R. Kiehl. *Etale cohomology and the Weil conjecture.* Springer-Verlag 1988.

[2] Laumon, G. and L. Moret-Bailly. *Champs algébriques.* Springer-Verlag, 1991.