Fibrations of Simplicial Sets (Lecture 9)

September 22, 2014

In the previous lecture, we gave a concrete criterion for a PL map of finite polyhedra $f : E \to B$ to be a fibration. Our goal in this lecture is to translate that criterion into a more combinatorial language.

Notation 1. Given a partially ordered set P, we let $\mathrm{N}(P)$ denote the nerve of P (regarded as a simplicial set): the n-simplices of $\mathrm{N}(P)$ are chains $a_0 \leq \cdots \leq a_n$ of elements of P.

Let X be a finite nonsingular simplicial set. We let $\Sigma(X)$ denote the collection of all nondegenerate simplices of X. We regard $\Sigma(X)$ as a partially ordered set by identifying each simplex σ with the simplicial subset of X given by the image of the map $\sigma : \Delta^n \to X$. We let $\mathrm{Sd}(X)$ denote the nerve of the partially ordered set $\Sigma(X)$. We will refer to $\mathrm{Sd}(X)$ as the *subdivision* of X.

Remark 2. Every map of nonsingular simplicial sets $f : X \to Y$ induces a map $\Sigma(X) \to \Sigma(Y)$, which carries each simplex $\sigma \subseteq X$ to its image in Y. We will denote this image by $f(\sigma)$. Beware that there is some potential for confusion here: if σ is an n-simplex of X, then $f(\sigma)$ need not be the image of σ under the induced map $X_n \to Y_n$ (since the latter image might be degenerate in Y).

Exercise 3. Let X be a finite nonsingular simplicial set. For each simplex $\sigma \in \Sigma(X)$, let v_σ be a point of $|X|$ which belongs to the interior of $|\sigma|$. Show that there is a unique homeomorphism $\alpha_X : |\mathrm{Sd}(X)| \to |X|$ which carries each vertex $\sigma \in \Sigma(X)$ to the point v_σ and carries each simplex of $|\mathrm{Sd}(X)|$ linearly to a simplex of $|X|$.

Remark 4. Let $f : X \to Y$ be a map of nonsingular simplicial sets. Suppose that we have chosen a vertex $v_\sigma \in |X|$ belonging to the interior of each simplex $\sigma \in \Sigma(X)$, and similarly we have chosen a vertex $w_\tau \in |Y|$ in the interior of each simplex $\tau \in \Sigma(Y)$. Suppose further that the v_σ and w_τ are *compatible* in the sense that the map $|f|$ carries each v_σ to the point $w_{f(\sigma)}$ (note that it is always possible to arrange this: in fact, if the points w_τ have been chosen we can always choose vertices v_σ which are compatible with them). Then the diagram of spaces

$$\begin{array}{ccc} |\mathrm{Sd}(X)| & \xrightarrow{\alpha_X} & |X| \\ \downarrow{\scriptstyle |\mathrm{Sd}(f)|} & & \downarrow{\scriptstyle |f|} \\ |\mathrm{Sd}(Y)| & \xrightarrow{\alpha_Y} & |Y| \end{array}$$

commutes.

Warning 5. For any nonsingular simplicial set X, we can make a canonical choice of the points v_σ by taking each v_σ to be the barycenter of $|\sigma|$. The homeomorphisms $\alpha_X : |\mathrm{Sd}(X)| \to |X|$ obtained in this way are functorial for embeddings of nonsingular simplicial sets, but not for general maps of nonsingular simplicial sets.

Definition 6. Let $f : X \to Y$ be a map of finite simplicial sets. We will say that f is *cell-like* if the induced map of topological spaces $|f| : |X| \to |Y|$ is cell-like.

Proposition 7. *Let $f : X \to Y$ be a map of finite nonsingular simplicial sets. The following conditions are equivalent:*

(1) *The map f is cell-like.*

(2) *For every simplex $\sigma \in \Sigma(Y)$, the partially ordered set $\{\tau \in \Sigma(X) : f(\tau) = \sigma\}$ has weakly contractible nerve.*

(3) *The induced map of partially ordered set $\Sigma(X) \to \Sigma(Y)$ is left cofinal: in other words, for every simplex $\sigma \in \Sigma(Y)$, the partially ordered set $\{\tau \in \Sigma(X) : \sigma \subseteq f(\tau)\}$ has weakly contractible nerve.*

Proof. Let σ be a nondegenerate simplex of Y, and fix a point v in the interior of $|\sigma|$. We can extend $\{v\}$ to a choice of one point in the interior of each simplex of Y, and then make compatible choices for X to obtain a commutative diagram

$$\begin{array}{ccc} |\operatorname{Sd}(X)| & \xrightarrow{\alpha_X} & |X| \\ \downarrow{\scriptstyle |\operatorname{Sd}(f)|} & & \downarrow{\scriptstyle |f|} \\ |\operatorname{Sd}(Y)| & \xrightarrow{\alpha_Y} & |Y|. \end{array}$$

It follows that the inverse image $|f|^{-1}\{v\}$ is homeomorphic to the geometric realization of the nerve of the partially ordered set $\{\tau \in \Sigma(X) : f(\tau) = \sigma\}$. Consequently, f is cell-like if and only if every such nerves is weakly contractible, which proves that (1) and (2) are equivalent. The equivalence of (2) and (3) follows from the fact that the inclusion of partially ordered sets

$$\{\tau \in \Sigma(X) : f(\tau) = \sigma\} \hookrightarrow \{\tau \in \Sigma(X) : \sigma \subseteq f(\tau)\}$$

has a right adjoint (given by forming the intersection with $f^{-1}\sigma$). □

We next make Proposition 7 more explicit for the nerve of a map of partially ordered sets.

Definition 8. Let $f : P \to Q$ be a map of partially ordered sets. We will say that f is a *Cartesian fibration* if, for every pair of elements $q' \leq q$ in Q and every element $p \in P$ with $f(p) = q$, the set $\{a \in P : a \leq p \text{ and } f(a) \leq q'\}$ has a largest element p' satisfying $f(p') = q'$.

In this case, the construction $p \mapsto p'$ determines a map of partially ordered sets $P_q = f^{-1}\{q\} \to f^{-1}\{q'\} = P_{q'}$. These maps are transitive in the evident sense, so that we can regard $q \mapsto P_q$ as a (contravariant) functor from Q to the category of partially ordered sets.

Conversely, suppose we are given a contravariant functor from Q to the category of partially ordered sets which assigns to each $q \in Q$ a partially ordered set P_q and to each $q' \leq q$ a map $\alpha_{q',q} : P_q \to P_{q'}$. We can then regard $P = \bigcup_{q \in Q} P_q$ as a partially ordered set by declaring $p' \leq p$ for $p' \in P_{q'}$ and $p \in P_q$ if and only if $q' \leq q$ and $p' \leq \alpha_{q',q} p$. Then the evident map $P \to Q$ is a Cartesian fibration of partially ordered sets.

Exercise 9. Check that the constructions outlined in Definition 8 are mutually inverse, so that the data of a partially ordered set P with a Cartesian fibration $P \to Q$ is equivalent to the data of a contravariant functor from Q to the category of partially ordered sets.

Example 10. Let $f : X \to Y$ be a map of nonsingular simplicial sets. Then the induced map $\Sigma(X) \to \Sigma(Y)$ is a Cartesian fibrations of partially ordered sets: for every simplex $\sigma \subseteq X$ and every facet $\tau \subseteq f(\sigma)$, the collection of those simplices of X which are contained in σ and lie over τ has a maximal element, given by $\tau \cap f^{-1}(\sigma)$.

If $f : P \to Q$ is a Cartesian fibration of simplicial sets, then the nerve $\mathrm{N}(P)$ can be identified with the homotopy colimit of the diagram $q \mapsto \mathrm{N}(P_q)$. In particular, if each fiber $\mathrm{N}(P_q)$ is weakly contractible, then the map $\mathrm{N}(P) \to \mathrm{N}(Q)$ is a weak homotopy equivalence. In fact, we can be more precise:

Corollary 11. *Let $f : P \to Q$ be a Cartesian fibration of partially ordered sets. The following conditions are equivalent:*

(1) *For each $q \in Q$, the fiber $P_q = f^{-1}\{q\}$ has weakly contractible nerve.*

(2) *The map of simplicial sets* $\mathrm{N}(P) \to \mathrm{N}(Q)$ *is cell-like.*

Proof. The implication (2) $\Rightarrow$ (1) is obvious (and does not require f to be a Cartesian fibration). Let us prove the converse. Let σ be a nondegenerate simplex of $\mathrm{N}(Q)$, given by a chain $\{q_0 < \cdots < q_m\}$. We let S_σ denote the collection of nondegenerate simplices of $\mathrm{N}(P)$ which lie over σ: that is, the collection of chains $\{p_0 < \cdots < p_n\}$ in P whose image in Q is σ. We wish to show that $\mathrm{N}(S_\sigma)$ is weakly contractible. Note that if $\sigma' \subseteq \sigma$, then the construction $\tau \mapsto \tau \cap f^{-1}\sigma'$ determines a map of partially ordered sets $S_\sigma \to S_{\sigma'}$. This map is a Cartesian fibration (exercise!)

Let us now fix a simplex $\sigma = \{q_0 < \cdots < q_m\}$. For $0 \leq i \leq m$, set $\sigma_i = \{q_i < q_{i+1} < \cdots < q_m\}$, so that we have a tower of Cartesian fibrations

$$S_\sigma = S_{\sigma_0} \to \cdots \to S_{\sigma_m}.$$

The simplicial set $\mathrm{N}(S_{\sigma_m}) \simeq \mathrm{Sd}(\mathrm{N}(P_{q_m}))$ is weakly contractible by virtue of assumption (1). To complete the proof, it will suffice to show that each of the maps $\theta : S_{\sigma_i} \to S_{\sigma_{i+1}}$ induces a weak homotopy equivalence of nerves. To prove this, it suffices to show that each fiber of θ has weakly contractible nerve. Fix a simplex $\tau = \{p_0 < \ldots < p_n\}$ in P lying over σ_{i+1}. Unwinding the definitions, we see that $\theta^{-1}\{\tau\}$ is the partially ordered set of nonempty chains in $R = \{p \in P : f(p) = q_i \text{ and } p \leq p_0\}$. That is, we have $\mathrm{N}(\theta^{-1}\{\tau\}) \simeq \mathrm{Sd}\,\mathrm{N}(R)$. It will therefore suffice to show that $\mathrm{N}(R)$ is weakly contractible. In fact, R has a largest element by virtue of our assumption that f is a Cartesian fibration. □

Remark 12. In the situation of Corollary 11, the assumption that f is a Cartesian fibration implies that the inclusion maps $f^{-1}\{q\} \hookrightarrow \{p \in P : q \leq f(p)\}$ admits a right adjoint, and therefore induces a weak homotopy equivalence. Consequently, the map $\mathrm{N}(P) \to \mathrm{N}(Q)$ is cell-like if and only if f is left cofinal.

We now ask when a map $f : X \to Y$ of finite nonsingular simplicial sets induces a fibration $|f| : |X| \to |Y|$. For every simplex $\sigma \in \Sigma(Y)$, let X_σ be the nerve of the partially ordered set $\{\tau \in \Sigma(X) : f(\tau) = \sigma\}$. Using a variation of the constructions sketched in the previous lecture, we see that $|X|$ can be identified with the homotopy colimit of the diagram $\sigma \mapsto |X_\sigma|$: that is, it is obtained by gluing together the products $|X_\sigma| \times |\sigma|$ (this is actually a slight generalization of the situation described in the previous lecture, because the simplices of X need not determine a triangulation of $|X|$: the intersection of two simplices need not be a simplex). Here the transition maps $X_\sigma \mapsto X_{\sigma'}$ are induced by maps of partially ordered sets given by $\tau \mapsto \tau \cap f^{-1}\{\sigma'\}$. Since these maps are Cartesian fibrations, we obtain the following:

Proposition 13. *Let $f : X \to Y$ be a map of finite nonsingular simplicial sets. The following conditions are equivalent:*

(a) *The map $|f| : |X| \to |Y|$ is a fibration.*

(b) *For every inclusion $\sigma' \subseteq \sigma$ in $\Sigma(Y)$ and every simplex $\tau' \in \Sigma(X)$ with $f(\tau') = \sigma'$, the partially ordered set $\{\tau \in \Sigma(X) : f(\tau) = \sigma \text{ and } \tau' = \tau \cap f^{-1}\sigma'\}$ has weakly contractible nerve.*

Remark 14. Condition (b) of Proposition 13 can be regarded as a combinatorial version of the path lifting property of fibrations for paths which begin in the interior of σ' and then enter the interior of σ.