Higher Simple Homotopy Theory (Lecture 7)

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Recall that finite polyhedra X and Y are *concordant* if there is a piecewise-linear fibration $q : E \to [0,1]$ with $X \simeq q^{-1}\{0\}$ and $Y \simeq q^{-1}\{1\}$. In the last lecture, we asserted that X and Y are simply homotopy equivalent if and only if they are concordant, and proved the "if" direction. Our goal in this lecture is to use this fact as a starting point for the study of "higher" simple homotopy theory, following ideas of Hatcher.

For any finite polyhedron B, we can contemplate piecewise-linear fibrations $q : E \to B$ (where E is also a finite polyhedron). Our first goal is to construct a universal example of such a fibration, so that the base B can be regarded as a classifying space for PL fibrations. It is not clear that such a classifying space exists in the setting of finite polyhedra, but we can give an *almost* tautological construction of one in the setting of simplicial sets.

Definition 1. For each integer n, let Δ^n denote the topological simplex of dimension n and let $\mathcal{M}_n$ denote the set of all finite polyhedra $E \subseteq \Delta^n \times \mathbb{R}^\infty$ for which the projection map $E \to \Delta^n$ is a fibration.

Note that for any linear map of simplices $\alpha : \Delta^m \to \Delta^n$, the construction $E \mapsto E \times_{\Delta^n} \Delta^m$ defines a map of sets $\alpha^* : \mathcal{M}_n \to \mathcal{M}_m$. In particular, we can regard the construction $[n] \mapsto \mathcal{M}_n$ as a simplicial set, which we will denote by $\mathcal{M}$.

Before we analyze the simplicial set $\mathcal{M}$, we need a few general facts about the relationship between polyhedra and simplicial sets.

Remark 2. Let K_0, K_1, and K_{01} be polyhedra, and suppose we are given piecewise linear embeddings

$$K_0 \overset{i_0}{\hookleftarrow} K_{01} \overset{i_1}{\hookrightarrow} K_1.$$

Then the pushout $K_0 \amalg_{K_{01}} K_1$ exists in the category of polyhedra: that is, we can regard endow $K_0 \amalg_{K_{01}} K_1$ with the structure of a polyhedron, where a map $K_0 \amalg_{K_{01}} K_1 \to L$ is piecewise linear if and only if its restriction to K_0 and K_1 is piecewise linear.

Beware that this need not be true if i_0 is not an embedding, even if i_1 is an embedding. This is often a technical nuisance.

Example 3. Let X be a finite simplicial set. We say that X is *nonsingular* if every simplex $\sigma : \Delta^n \to X$ is either degenerate (meaning that it factors through Δ^m for $m < n$) or is a monomorphism of simplicial sets (in particular, all the faces of σ are again nondegenerate).

For any nonsingular finite simplicial set X, the geometric realization $|X|$ can be regarded as a finite polyhedron. More precisely, there is a unique PL structure on $|X|$ having the property that for every nondegenerate n-simplex of X, the associated map $\Delta^n \to |X|$ is piecewise linear (this follows by invoking Remark 2 repeatedly).

In what follows, we will often not distinguish between a (finite nonsingular) simplicial set X and the polyhedron $|X|$. For example, we use the symbol Δ^n to denote both the n-simplex as a simplicial set and the topological n-simplex, and apply similar considerations to the boundary $\partial \Delta^n$ and the horns $\Lambda^n_i \subseteq \Delta^n$.

We will also need the following technical fact, whose proof we omit (see Lemma 2.7.12 of [1]):

Proposition 4. *Let $q : E \to B$ be a map of finite polyhedra. The following conditions are equivalent:*

(1) *The map q is a fibration.*

(2) *For every triangulation of B and every simplex σ of the triangulation, the induced map $E \times_B \sigma \to \sigma$ is a fibration.*

(3) *There exists a triangulation of B such that, for every simplex σ of the triangulation, the induced map $E \times_B \sigma \to \sigma$ is a fibration.*

Corollary 5. *Let B be a finite nonsingular simplicial set. Then* $\mathrm{Hom}(B, \mathcal{M})$ *can be identified with the set of finite polyhedra $E \subseteq |B| \times \mathbb{R}^\infty$ for which the projection map $E \to |B|$ is a fibration.*

Proof. The geometric realization $|B|$ admits a triangulation for which each simplex is contained in the image of some simplex of B (beware that the nondegenerate simplices of B do not generally themselves determine a triangulation of $|B|$, unless one is liberal with the meaning of the word "triangulation"). ☐

Corollary 6. *The simplicial set $\mathcal{M}$ is a Kan complex.*

Proof. Suppose we are given a map $f_0 : \Lambda^n_i \to \mathcal{M}$, given by a polyhedron $E \subseteq |\Lambda^n_i| \times \mathbb{R}^\infty$ for which the projection $E \to |\Lambda^n_i|$ is a fibration. Choose a piecewise linear retraction $r : |\Delta^n| \to |\Lambda^n_i|$, and define $\overline{E} = E \times_{|\Lambda^n_i|} |\Delta^n|$. Then $\overline{E}$ can be identified with a map $f : \Delta^n \to \mathcal{M}$ extending f_0. ☐

We next investigate the role of the Kan complex $\mathcal{M}$ as a "classifying space."

Exercise 7. Let B be a finite polyhedron. Suppose we are given fibrations of finite polyhedra $f : X \to B$, $g : Y \to B$. We will say that f and g are *concordant* if there exists a fibration of finite polyhedra $h : Z \to B \times [0, 1]$ for which the inverse image of $B \times \{0\}$ is isomorphic to X and the inverse image of $B \times \{1\}$ is isomorphic to Y. Show that concordance is an equivalence relation.

Let B be a finite nonsingular simplicial set. Any map $f : B \to \mathcal{M}$ determines a fibration of finite polyhedra $E_f \to |B|$, and any homotopy between maps $f, g : |B| \to \mathcal{M}$ determines a concordance from E_f to E_g. We therefore obtain a well-defined map from the set $[B, \mathcal{M}]$ of homotopy classes of maps from B into $\mathcal{M}$ to the set of concordance classes of fibrations over $|B|$.

Proposition 8. *This map is bijective.*

Proof. To prove surjectivity, it suffices to note that for any map of finite polyhedra $X \to |B|$, we can choose a compatible PL embedding of X into $|B| \times \mathbb{R}^\infty$.

To prove injectivity, it suffices to show that if $X \subseteq |B| \times \mathbb{R}^\infty$ and $Y \subseteq |B| \times \mathbb{R}^\infty$ are polyhedra fibered over $|B|$ and we are given any concordance $Z \to |B \times \Delta^1|$ from X to Y, then we can choose a PL embedding of Z into $|B \times \Delta^1| \times \mathbb{R}^\infty$ which is compatible with the given embeddings on X and Y. ☐

References

[1] Waldhausen, F., Jahren, B. and J. Rognes. *Spaces of PL Manifolds and Categories of Simple Maps.*