Whitehead Torsion, Part II (Lecture 4)

September 9, 2014

In this lecture, we will continue our discussion of the Whitehead torsion of a homotopy equivalence $f : X \to Y$ between finite CW complexes. In the previous lecture, we gave a definition in the special case where f is cellular. To remove this hypothesis, we need the following:

Proposition 1. *Let X and Y be connected finite CW complexes and suppose we are given cellular homotopy equivalences $f, g : X \to Y$. If f and g are homotopic, then $\tau(f) = \tau(g) \in \mathrm{Wh}(\pi_1 X)$.*

Lemma 2. *Suppose we are given quasi-isomorphisms $f : (X_*, d) \to (Y_*, d)$ and $g : (Y_*, d) \to (Z_*, d)$ between finite based complexes with $\chi(X_*, d) = \chi(Y_*, d) = \chi(Z_*, d)$. Then*

$$\tau(g \circ f) = \tau(g)\tau(f)$$

in $\widetilde{K}_1(R)$.

Proof. We define a based chain complex (W_*, d) by the formula

$$W_* = X_{*-1} \oplus Y_* \oplus Y_{*-1} \oplus Z_*$$

$$d(x, y, y', z) = (-dx, f(x) + dy + y', -dy', g(y') + dz).$$

Then (W_*, d) contains $(C(f)_*, d)$ as a based subcomplex with quotient $(C(g)_*, d)$, so an Exercise from the previous lecture gives $\tau(W_*, d) = \tau(g)\tau(f)$. We now choose a new basis for each W_* by replacing each basis element of $y \in Y_*$ by $(0, y, 0, g(y))$; this is an upper triangular change of coordinates and therefore does not affect the torsion $\tau(W_*, d)$. Now the construction $(y', y) \mapsto (0, y, y', g(y))$ identifies $C(\mathrm{id}_Y)_*$ with a based subcomplex of W_* having quotient $C(-g \circ f)_*$. Applying the same Exercise again we get

$$\tau(W_*, d) = \tau(\mathrm{id}_Y)\tau(-g \circ f) = \tau(g \circ f).$$

□

Remark 3. Suppose that $f : X \to Y$ is the inclusion of X as a subcomplex of Y. Let $\lambda : C_*(\widetilde{X}; \mathbf{Z}) \to C_*(\widetilde{Y}; \mathbf{Z})$ be as above. Then the mapping cone $C(\lambda)_*$ contains the mapping cone $C(\mathrm{id}_{C_*(\widetilde{X};|Z|)})_*$ as a based subcomplex, and the quotient is the relative cellular chain complex $C_*(\widetilde{Y}, \widetilde{X}; \mathbf{Z})$. It follows that the Whitehead torsion of f can be computed as (the image in $\mathrm{Wh}(\pi_1 X)$ of the torsion of the acyclic complex $C_*(\widetilde{Y}, \widetilde{X}; \mathbf{Z})$.

Proof of Proposition 1. Choose a homotopy $h : X \times [0, 1] \to Y$ from $f = h_0$ to $g = h_1$. We may assume without loss of generality that h is cellular. Then the Whitehead torsion $\tau(h)$ is well-defined; we will prove that $\tau(f) = \tau(h) = \tau(g)$. Note that f is given by the composition

$$X \times \{0\} \overset{i}{\hookrightarrow} X \times [0, 1] \overset{h}{\to} Y.$$

Then $\tau(f) = \tau(h)\tau(i)$ in $\mathrm{Wh}(\pi_1 X)$ (Lemma 2). It will therefore suffice to show that $\tau(i)$ vanishes. Using Remark 3, we can identify $\tau(i)$ with the torsion of the relative cellular chain complex

$$C_*(\widetilde{X} \times [0,1], \widetilde{X} \times \{0\}; \mathbf{Z}),$$

which vanishes (we saw this in the previous lecture). □

If $f : X \to Y$ is *any* homotopy equivalence between connected finite CW complexes, we define $\tau(f) = \tau(f_0)$ where f_0 is a cellular map which is homotopic to f. By virtue of Proposition 1, this definition is independent of the choice of f_0.

Proposition 4. *Let $f : X \to Y$ and $g : Y \to Z$ be homotopy equivalences between connected finite CW complexes, all having fundamental group G. Then $\tau(gf) = \tau(g)\tau(f)$ in $\mathrm{Wh}(G)$.*

Proof. This follows immediately from Lemma 2. □

Corollary 5. *Let $f : X \to Y$ be a simple homotopy equivalence between finite CW complexes. Then $\tau(f) = 1$.*

Proof. Using Proposition 4, we can reduce to the case where f is an elementary expansion. In this case, $\tau(f)$ is the torsion of the relative cellular chain complex $C_*(\widetilde{Y}, \widetilde{X}; \mathbf{Z}[G])$ which has the form

$$\cdots \to 0 \to \mathbf{Z}[G] \stackrel{\pm g}{\to} \mathbf{Z}[G] \to 0.$$

□

Remark 6. Let X be a finite connected CW complex and set $G = \pi_1 X$. Then every element $\eta \in \mathrm{Wh}(G)$ can be realized as the Whitehead torsion of a homotopy equivalence $f : X \to Y$. To see this, choose any matrix $M \in \mathrm{GL}_n(\mathbf{Z}[G])$. Fix an integer $k \geq 2$ and let X' be the CW complex obtained from X by attaching n copies of S^k at some base point $x \in X$, so we have an evident retraction $r : X' \to X$. Applying the relative Hurewicz theorem to the map of universal covers $\widetilde{X}' \to \widetilde{X}$, we obtain a canonical isomorphism $\pi_{k+1}(X, X') \simeq \mathbf{Z}[G]^n$. Consequently, the matrix M provides the data for attaching n copies of D^{k+1} to X' in such a way that that the retraction r extends over the resulting CW complex Y. The inclusion $X \hookrightarrow Y$ is an isomorphism on fundemental groups, and the relative chain complex of the inclusion of universal covers is given by

$$\cdots \to 0 \to \mathbf{Z}[G]^n \stackrel{M}{\to} \mathbf{Z}[G]^n \to 0 \to \cdots.$$

Since M is invertible, we conclude that the inclusion $f : X \to Y$ is a homotopy equivalence and that $\tau(f) \in \mathrm{Wh}(G)$ is represented by the matrix $M^{\pm 1}$ (depending on the parity of k)

The Whitehead groups $\mathrm{Wh}(G)$ are generally nonzero:

Example 7. Let G be an abelian group. Then the determinant homomorphism $K_1(\mathbf{Z}[G]) \to \mathbf{Z}[G]^\times$, which induces a surjective map

$$\mathrm{Wh}(G) \to (\mathbf{Z}[G]^\times)/\{\pm g\}_{g \in G}.$$

The group on the right generally does not vanish. For example, if $G = \mathbf{Z}/5\mathbf{Z}$, then $\mathbf{Z}[G] \simeq \mathbf{Z}[t]/(t^5 - 1)$ contains a unit $1 - t^2 - t^3$ (with inverse $1 - t - t^4$) which is not of the form $\pm t^i$.

Combined with Remark 6, this supplies a negative answer to the question raised in the previous lecture: there exist homotopy equivalences between finite CW complexes with nonvanishing torsion, and such homotopy equivalences cannot be simple. However, it turns out that the Whitehead torsion is the only obstruction:

Theorem 8 (Whitehead)**.** *Let $f : X \to Y$ be a homotopy equivalence between connected finite CW complexes with $\tau(f) = 1 \in \mathrm{Wh}(G)$, where $G = \pi_1 X$. Then f is a simple homotopy equivalence.*

Example 9. One can show that the determinant map $K_1(\mathbf{Z}) \to \mathbf{Z}^\times = \{\pm 1\}$ is an isomorphism, so that the Whitehead group $\mathrm{Wh}(G)$ vanishes when G is the trivial group. Theorem 8 then implies that any homotopy equivalence between *simply connected* finite CW complexes is a simple homotopy equivalence.

Example 10. A nontrivial theorem of Bass, Heller, and Swan asserts that the Whitehead group $\mathrm{Wh}(\mathbf{Z}^d)$ is trivial for each integer d. Together with the s-cobordism theorem, this implies that every h-cobordism from a torus T^d to another manifold M is isomorphic to a product $T^d \times [0,1]$.

For use in the proof of Theorem 8, we include the following example of a simple homotopy equivalence:

Example 11. Let X be a finite CW complex, and suppose we are given a pair of maps

$$f, g : S^{n-1} \to X^{n-1}.$$

Let Y and Z be the CW complexes obtained from X by attaching n-cells along f and g, respectively. Then Y and Z are simple homotopy equivalent. To see this, choose a homotopy $h : S^{n-1} \times [0,1] \to X^n$, and let W be the cell complex obtained from $Y \amalg_X Z$ by attaching an $(n+1)$-cell along the induced map

$$D^n \amalg_{S^{n-1}\times\{0\}} (S^{n-1} \times [0,1]) \amalg_{S^{n-1}\times\{1\}} D^n \to Y \amalg_X Z.$$

Then the inclusions $Y \hookrightarrow W \hookleftarrow Z$ are both elementary expansions.

Let us conclude this lecture by sketching a proof of Theorem 8. Let $f : X \to Y$ be a homotopy equivalence of finite CW complexes such that $\tau(f) = 1$; we wish to show that f is a simple homotopy equivalence. Without loss of generality we may assume f is cellular. Replacing Y by the mapping cylinder $M(f)$, we can assume that f is the inclusion of a subcomplex.

Fix a cell e of minimal possible dimension which belongs to Y but not to X; we will regard this cell as the image of a map g from a hemisphere S^n_- into Y which carries the equator $S^{n-1} \subseteq S^n_-$ into X^{n-1}. Since the inclusion f is a homotopy equivalence, the map g is homotopic to a map from the disk into X via a homotopy which is fixed on S^{n-1}; we may regard this homotopy as defining a map $\overline{g} : D^{n+1} \to Y$ carrying S^n_+ into X. Let us identify D^{n+1} with the lower hemisphere S^{n+1}_- of an $(n+1)$-sphere S^{n+1}, and let Y' denote the elementary expansion of Y given by $Y \amalg_{S^{n+1}_-} D^{n+2}$; we will denote the interior of D^{n+2} by $e' \subseteq Y'$.

Let X' be the subcomplex of Y' given by the union of X and the upper hemisphere S^{n+1}_+. Then X' is an elementary expansion of X. The cells of Y' that do not belong to X' are almost exactly the same as the cells of Y that do not belong to X: the only exception is that Y' has a new cell e' of dimension $n+2$, and that the cell $e \subseteq Y$ now belongs to X'. Replacing the inclusion $X \hookrightarrow Y$ by $X' \hookrightarrow Y'$, we have "traded up" an n-cell for an $(n+2)$-cell. Repeating this process finitely many times, we can reduce to the case where Y is obtained from X by adding only cells of dimension n and $n+1$ for some $n \geq 2$. Let us denote the cells of dimension n by $e_1, \ldots, e_m$ and the cells of dimension $(n+1)$ by $e'_1, \ldots, e'_m$ (note that the number of $(n+1)$-cells is necessarily equal to the number of n-cells, since f is a homotopy equivalence).

Let Y_0 be the subcomplex of Y obtained from X by attaching only the n-cells. We have a long exact sequence

$$\pi_{n+1}(Y, X) \to \pi_{n+1}(Y, Y_0) \xrightarrow{M} \pi_n(Y_0, X) \to \pi_n(Y, X).$$

Since the inclusion $X \hookrightarrow Y$ is a homotopy equivalence, the groups $\pi_n(Y, X)$ and $\pi_{n+1}(Y, X)$ are trivial, so that M is an isomorphism. Using the relative Hurewicz theorem, we see that $\pi_n(Y_0, X) \simeq \mathrm{H}_n(Y_0, X; \mathbf{Z}[G])$ is a free module $\mathbf{Z}[G]^m$. Moreover, it *almost* has a canonical basis: each of the cells e_i determines a generator of $\pi_n(Y_0, X)$ which is ambiguous up to a sign (due to orientation issues) and to the action of G (due to base point issues). Similarly, the cells e'_i determine a basis for $\pi_{n+1}(Y, Y_0) \simeq \mathbf{Z}[G]^m$ which is ambiguous up to the action of $\pm G$. Then M can be regarded as an element of $\mathrm{GL}_m(\mathbf{Z}[G])$, and the image of M in $\mathrm{Wh}(G)$ is given by $\tau(f)^{\pm 1}$ (where the sign depends on the parity of n). Since $\tau(f) = 1$, it is possible to choose bases as above so that M belongs to the commutator subgroup of $\mathrm{GL}_m(\mathbf{Z}[G])$.

Let $x \in X$ be a base point. Let Y^+ denote the elementary expansion

$$Y^+ = ((Y \amalg_{\{x\}} D^{n+1}) \amalg_{\{x\}} D^{n+1}) \amalg \cdots$$

obtained from Y by attaching m copies of the disk D^{n+1} at the base point of X; here we regard Y^+ as obtained from Y by adding m cells of dimension n (the boundaries of the new disks) and m cells of dimension $(n+1)$ (the interiors of the new disks). Replacing Y by Y^+ has the effect of replacing m by $2m$ and M by the matrix

$$M^+ = \begin{bmatrix} M & 0 \\ 0 & \mathrm{id} \end{bmatrix}.$$

Since M belongs to the commutator subgroup of $\mathrm{GL}_m(\mathbf{Z}[G])$, the matrix M^+ can be written as a product of matrices of the form

$$\begin{bmatrix} \mathrm{id} & X \\ 0 & \mathrm{id} \end{bmatrix} \text{ or } \begin{bmatrix} \mathrm{id} & 0 \\ Y & \mathrm{id} \end{bmatrix}$$

(see Remark 12 below). Replacing Y by Y^+, m by $2m$, and M by M^+, we may reduce to the case where M has the form

$$M_1 \cdots M_k$$

where each M_i is either upper-triangular or lower-triangular.

To complete the proof, it will suffice to show that for a homotopy equivalence $f : X \to Y$ with associated matrix M as above and any matrix U which is either upper (or lower) triangular, we can find another homotopy equivalence $f' : X \to Y'$ with associated matrix MU for which the induced homotopy equivalence $Y \simeq Y'$ is simple. To see this, consider the filtration

$$Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_m = Y$$

where $Y_i = Y_0 \cup e'_1 \cup \cdots \cup e'_i$. Using Example 11, we see that the simple homotopy type of Y_i is unchanged if we modify the attaching map $\partial\, e'_i \to Y_0$ by an arbitrary homotopy within Y_{i-1}; by means of such modifications we can multiply M by any upper-triangular matrix that we like.

Remark 12 (Whitehead's Lemma)**.** Let R be any ring, and let $H \subseteq \mathrm{GL}_2(R)$ be the subgroup generated by matrices of the form

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}.$$

For every invertible element $g \in R$, the calculation

$$\begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -g^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & g \\ -g^{-1} & 0 \end{bmatrix}$$

shows that $\begin{bmatrix} 0 & g \\ -g^{-1} & 0 \end{bmatrix} \in H$.

If g and h are invertible elements of R, we have

$$\begin{aligned} \begin{bmatrix} ghg^{-1}h^{-1} & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & g \\ -g^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & h^{-1} \\ -h & 0 \end{bmatrix} \begin{bmatrix} 0 & (hg)^{-1} \\ -hg & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &\in H. \end{aligned}$$

Replacing R by the ring of n-by-n matrices over R, we see that any element of the commutator subgroup of $\mathrm{GL}_n(R)$ can be written (in $\mathrm{GL}_{2n}(R)$) as a product of matrices of the form

$$\begin{bmatrix} \mathrm{id} & X \\ 0 & \mathrm{id} \end{bmatrix} \text{ and } \begin{bmatrix} \mathrm{id} & 0 \\ Y & \mathrm{id} \end{bmatrix}.$$