# Thickenings of a Point (Lecture 38)

March 4, 2015

Let $K$ be a finite polyhedron. Recall that an *$n$-manifold thickening* of $K$ is a PL map $\pi : M \to K$, where $M$ is a compact PL $n$-manifold equipped with a stable framing having the property that the fibers of $\pi$ are contractible and the fibers of $\pi|_{\partial M}$ are simply connected. We let $T^n(K)$ denote the classifying space of $n$-manifold thickenings of $K$ introduced in Lecture 35. Our goal, in this lecture and the next, is to show that $T^n(K)$ is highly connected when $n$ is large compared with the dimension of $K$. We begin by treating the simplest case.

**Proposition 1.** *If $n \geq 6$, then the space $T^n(*)$ is $(n-1)$-connected.*

Note that $T^n(*)$ is a classifying space for stably framed PL $n$-manifolds $M$ for which $M$ is contractible and $\partial M$ is simply connected. Let $M$ be such a manifold, and let $x$ be any point in the interior of $M$, and let $N \subseteq M$ be the manifold obtained from $M$ by removing the interior of a small disk $D$ around $x$. Then we can regard $N$ as a bordism from $\partial D \simeq S^{n-1}$ to $\partial M$. We have $\mathrm{H}_*(N, \partial D) \simeq \mathrm{H}_*(M, D) \simeq 0$ by virtue of our assumption that $M$ is contractible, and $\mathrm{H}_*(N, \partial M) \simeq \mathrm{H}^{n-*}(N, \partial D) \simeq \mathrm{H}^{n-*}(M, D) \simeq 0$ using Poincare duality. Consequently, the inclusions

$$\partial D \hookrightarrow N \hookleftarrow \partial M$$

are homology equivalences. Since all the spaces involved are simply connected, they are also homotopy equivalences. Since $n \geq 6$, it follows from the (PL) h-cobordism theorem that $N$ is PL homeomorphic to a product $\partial D \times [0, 1]$, so that $M$ is PL-homeomorphic to a disk. We can summarize the situation by saying that for $n \geq 6$, we can regard $T^n(*)$ as a classifying space for stably framed PL disks of dimension $n$.

In what follows, it will be more convenient to work instead with topological disks: by virtue of Kirby-Siebenmann theory, this makes no difference (the classifying space of stably framed topological disks of dimension $n \geq 6$ is homotopy equivalent to the classifying space of stably framed PL disks of dimension $n \geq 6$, since the classification of PL structures is governed by an h-principle). For every topological manifold $M$, let $\mathrm{Top}(M)$ denote the homeomorphism group of $M$. Restricting homeomorphisms to the interior supplies a map $\mathrm{Top}(D^n) \to \mathrm{Top}(\mathbb{R}^n)$, and we have stabilization maps $\mathrm{Top}(\mathbb{R}^n) \to \mathrm{Top}(\mathbb{R}^{n+1}) \to \cdots$ with colimit Top. Unwinding the definitions, we can identify $T^n(*)$ with the homotopy fiber of the composite map

$$\mathrm{BTop}(D^n) \to \mathrm{BTop}(\mathbb{R}^n) \to \mathrm{BTop}\,.$$

Proposition 1 is therefore a consequence of the following two results:

**Proposition 2.** *The map $\mathrm{BTop}(D^n) \to \mathrm{BTop}(\mathbb{R}^n)$ has $n$-connected homotopy fibers.*

**Proposition 3.** *The stabilization map $\mathrm{BTop}(\mathbb{R}^n) \to \mathrm{BTop}(\mathbb{R}^{n+1})$ has $(n-1)$-connected homotopy fibers.*

Let us begin by analyzing the assertion of Proposition 3. If $X$ is a locally compact topological space, let $X^+$ denote its one-point compactification, and let $\infty$ denote the "point at infinity" of $X^+$. The construction $X \mapsto X^+$ is functorial for homeomorphisms, and induces a map $\mathrm{Top}(X) \to \mathrm{Top}(X^+)$ whose image is the subgroup $\mathrm{Top}(X^+, \infty) \subseteq \mathrm{Top}(X^+)$ consisting of homeomorphisms which fix the point at infinity. When $X = \mathbb{R}^n$, the homeomorphism group of $X^+$ acts transitively on $X^+$, so we that the homotopy fiber of the induced map $\mathrm{BTop}(X) \to \mathrm{BTop}(X^+)$ can be identified with $\mathrm{Top}(X^+)/\mathrm{Top}(X) \simeq X^+$. In particular, the map

$$\mathrm{BTop}(R^n) \to \mathrm{BTop}(S^n)$$

can be regarded as a fibration (up to homotopy) whose fibers are $n$-dimensional spheres.

**Proposition 4** (Alexander Trick)**.** *Restriction to the boundary induces a homotopy equivalence* $\mathrm{Top}(D^{n+1}) \to \mathrm{Top}(S^n)$.

*Proof.* Let us identify $D^{n+1}$ with the unit ball in $\mathbb{R}^{n+1}$, and $S^n$ with its boundary. Every homeomorphism $h : S^n \to S^n$ can extended "radially" to a homoemorphism $\widehat{h} : D^{n+1} \to D^{n+1}$, given by the formula

$$\widehat{h}(x) = \begin{cases} 0 & \text{if } x = 0 \\ |x|h(\frac{x}{|x|}) & \text{if } x \neq 0. \end{cases}$$

The construction $h \mapsto \widehat{h}$ determines a section of the restriction map $\mathrm{Top}(D^{n+1}) \to \mathrm{Top}(S^n)$. We claim that this section is also a homotopy inverse. To prove this, we note that if $H : D^{n+1} \to D^{n+1}$ is an arbitrary homeomorphism and $h = H|_{S^n}$, then $\widehat{h}$ and $H$ are related by a canonical isotopy $\{f_t\}_{0 \leq t \leq 1}$, given by the formula

$$f_t(x) = \begin{cases} |x|h(\frac{x}{|x|} & \text{if } t \leq |x|, x \neq 0 \\ tH(\frac{x}{t}) & \text{if } |x| < t \\ 0 & \text{if } t = x = 0. \end{cases}$$

□

The resulting homotopy equivalence $\mathrm{BTop}(D^{n+1}) \simeq \mathrm{BTop}(S^n)$ means that, for any reasonable space $B$, there is a bijective correspondence between equivalence classes of fiber bundles of closed $(n+1)$-disks over $B$ and fiber bundles of $n$-spheres over $B$. In one direction, this is given by passing to the boundary; in the other, it is given by forming the cone.

Let $\phi$ denote the composition

$$\mathrm{BTop}(\mathbb{R}^n) \simeq \mathrm{BTop}(S^n, \infty) \to \mathrm{BTop}(S^n) \simeq \mathrm{BTop}(D^{n+1})$$

and let $\psi : \mathrm{BTop}(D^{n+1}) \to \mathrm{BTop}(\mathbb{R}^{n+1})$ be given by restriction to the interior. Then:

**Proposition 5.** *The composition*

$$\mathrm{BTop}(\mathbb{R}^n) \xrightarrow{\phi} \mathrm{BTop}(D^{n+1}) \xrightarrow{\psi} \mathrm{BTop}(\mathbb{R}^{n+1})$$

*is homotopic to the stabilization map.*

*Proof.* Let $E \to B$ be an $\mathbb{R}^n$-bundle over some (reasonably nice) space $B$. Applying $\psi \circ \phi$, we obtain an $\mathbb{R}^{n+1}$-bundle over $B$. The latter bundle can be described as follows: first, we pass to an $S^n$-bundle $\widehat{E}$ by taking the one-point compactification fiberwise. Then we write $\widehat{E}$ as the boundary of a $D^{n+1}$-bundle $F$ over $B$, and then take the (fiberwise) interior. We can construct $F$ explicitly by applying fiberwise one-point compactification to the bundle $E \times \mathbb{R}_{\geq 0}$, in which case the interior of $F$ is given by $E \times \mathbb{R}_{>0} \simeq E \times \mathbb{R}$. □

Note that the map $\phi$ is homotopy equivalent to the map $\mathrm{BTop}(S^n, \infty) \to \mathrm{BTop}(S^n)$, which is an $S^n$-bundle and therefore has $(n-1)$-connected homotopy fibers. Consequently, to prove that the stabilization map $\mathrm{BTop}(\mathbb{R}^n) \to \mathrm{BTop}(\mathbb{R}^{n+1})$ has $(n-1)$-connected homotopy fibers, it will suffice to show that the map $\psi$ has $(n-1)$-connected homotopy fibers. In other words, Proposition 3 is a consequence of Proposition 2. Let us therefore concentrate on the latter.

The above analysis yields a fiber sequence

$$S^{n-1} \to \mathrm{Top}(\mathbb{R}^n)/\mathrm{Top}(\mathbb{R}^{n-1}) \to \mathrm{Top}(\mathbb{R}^n)/\mathrm{Top}(D^n),$$

where the second map is surjective. Consequently, to show that $\mathrm{Top}(\mathbb{R}^n)/\mathrm{Top}(D^n)$ is $n$-connected, it will suffice to show that the homotopy fibers of the map $S^{n-1} \to \mathrm{Top}(\mathbb{R}^n)/\mathrm{Top}(\mathbb{R}^{n-1})$ are $(n-1)$-connected. Let us identify $S^{n-1}$ with the quotient $O(n)/O(n-1)$ and, using Kirby-Siebenmann theory, let us identify $\mathrm{Top}(\mathbb{R}^n)/\mathrm{Top}(\mathbb{R}^{n-1})$ with $\mathrm{PL}(n)/\mathrm{PL}(n-1)$ (remember that $n \geq 6$). We wish to show that the natural map

$$O(n)/O(n-1) \to \mathrm{PL}(n)/\mathrm{PL}(n-1)$$

has $(n-1)$-connected homotopy fibers. Note that these homotopy fibers can be identified with total homotopy fibers of the diagram

$$\begin{array}{ccc} \mathrm{BO}(n-1) & \longrightarrow & \mathrm{BO}(n) \\ \downarrow & & \downarrow \\ \mathrm{BPL}(n-1) & \longrightarrow & \mathrm{BPL}(n), \end{array}$$

which are also homotopy fibers of the stabilization map

$$\sigma : \mathrm{PL}(n-1)/O(n-1) \to \mathrm{PL}(n)/O(n)$$

We are therefore reduced to proving that $\sigma$ has $(n-1)$-connected homotopy fibers. This is an equivalent formulation of the main result of [2] (see also [1] for an informal exposition).

## References

[1] Lurie, J. Lecture notes for 18.937 (Lectures 18-22), available on my webpage.

[2] Hirsch, M. and B. Mazur. *Smoothings of Piecewise Linear Manifolds.*

[3] Waldhausen, F., B. Jahren and J. Rognes. *Spaces of PL Manifolds and Categories of Simple Maps.*